A STUDY OF CATEGORIAL STRUCTURES

Thesis
Submitted for the Award of the Degree
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IN
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BY

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DEDICATED TO

MY MENTOR

PROFESSOR M. A. KAZIM

رَبِّ زِدْنِي عِلْمًاہ

"My Lord ! Increase me in Knowledge"

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CERTIFICATE

This is to certify that the contents of this thesis entitled "A study of Categorial Structures" is the original research work of Mr. Shabbir Khan carried out under my supervision. He has fulfilled the prescribed conditions given in the ordinances and regulations of the Aligarh Muslim University, Aligarh.

I further certify that the work has not been submitted either partly or fully to any other university or institution for the award of any degree.

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PREFACE

The present thesis entitled "A STUDY OF CATEGORIAL STRUCTURES", is a part of my research work carried out in the Department of Mathematics, Aligarh Muslim University, Aligarh under the thought-provoking and constant guidance of Dr. S.M.A. Zaidi.

The objective of this work is to study the concepts of BCK and BCI-algebras in terms of categorial language. Although some initiatives have been taken by K. Iseki, Hiroshi Yutani and Axel Mobus, but till now there is no complete literature about the categories of BCK and BCI-algebras. In this dissertation I have concentrated on the categories of BCK and BCI-algebras and tried to ~~fulfil~~ fill ~~a~~ the wide gap in the subject.

The Thesis comprises five chapters followed by a bibliography. The first chapter is introductory and is basically intended to make the thesis as self-contained as possible. For the convenience of ready references a general introduction of the fundamental notions of BCK and BCI-algebras and relevant results to be used in subsequent chapters have been collected in this chapter. For the categorial notion we refer to [3], [4], [5], [11] and [28].

Chapter II is devoted to the formation and observations of different types BCK and BCI-algebraic categories. This chapter is developed on the basis of Blyth [3]. Criteria of mono and epimorphism in BCK (BCI)-algebraic categories have been studied. [Ths 2.1 & 2.2], sub-object, quotient object and zero morphisms are discussed in the categories of BCK and BCI-algebras [Prop. 2.1 & 2.2]. Categorial structures products and sums [props. 2.3 to 2.4], Kernels, Co-Kernels, images and co-images [Prop 2.6 to 2.11], equalizers and Co-equalizers [Ths. 2.6 & 2.7] and intersections, pullbacks, pushouts [Ths. 2.8 to 2.10] have been studied in these categories.

Chapter III investigates some structural properties of different BCK and BCI-algebraic categories. This chapter has been developed based on the results of Blyth [3] and Freyd [11]. Unique factorization, normality and co-normality have been checked [Ths. 3.1 to 3.3]. Further the category of BCK-algebras and normal homomorphisms has been tested for unique factorization, binormality and exactness criteria [Thes 3.4 to 3.8]. Structural properties such as existence of Kernels, Co-kernels, Products, sums, balanceness, exactness and abelianization have been checked [Ths 3.9 to 3.16].

Chapter IV is concerned with the study of BCK(BCI)-homomorphisms through well defined homological techniques. Characterization of regular homomorphism, existence of first and second isomorphism theorems is observed in the category of BCK-algebras [Ths. 4.1 to 4.3]. The characterization of different BCK(BCI)-homorphisms in different BCK(BCI)-algebraic categories has been described [Ths. 4.4 to 4.9]. In section 4.3 some homological results on filters and prime ideals in the category of BCK-algebras have been proved [Props. 4.1-4.6]. The concept of commutator ideal of a BCK-algebra is introduced and some basic results are given [Props. 4.7 to 4.8], some useful results on commutators have been proved to check the functorial properties of BCK-algebras [Prop 4.10, cors 4.3 & 4.4]. However, characterization of commutative BCK-algebra is described [Th 4.11 and Cor. 4.4] Further Th 4.12, Corollaries 4.5 and 4.6 yield functorial property of BCK-algebras through commutator ideals.

The last chapter of the thesis describes the functorial properties of some well defined BCK and BCI-structures. Several functors have been constructed through different BCK (BCI)-algebraic categories. Functors with the help of p-radical property of BCI-algebras are constructed [Props. 5.1, 52, Cor. 5.2 and lemma 5.4]. In section 5.3 functors by retractions and co-retractions are constructed [Lemna 5.5, 5.6, Props. 5.4 to 5.9 and Cor. 5.3 to 5.4]. Functor by self maps is constructed in section 5.5. A functor by involution of a bounded commutative BCK-algebra is given in section 5.6. Hom functors in the categories of BCK and BCI-algebras are described in section 5.7. Using the p-semi-simple property of a BCI-algebra some functors have been constructed [Props

5.11] and Lemmas 5.8 & 5.9]. In the end functors by commutators and natural isomorphisms have been studied.

Out of the Chapter III a research paper entitled "A note on the category of p-semi-simple algebras communicated for publication to the Soochow Jr. Math; Soochow Univ., Taipei, Taiwan, R.O.C.

While a paper based on some results from Chapter IV has been communicated for publication to the Aligarh Bull. Math. Another paper based on certain results of Chapters IV and V combined has been sent for publication to the Radovi, Mathematicki, Yugoslavia.

It has been my privilege indeed to work under the supervision of my reverend teacher Dr. S.M.A. Zaidi. His rigorous training and constant encouragement were solely instrumental in getting the work completed. In fact his useful suggestions and careful guidance resulted in a lot of improvements of the contents. I take this opportunity to put on record my profound feelings of indebtedness to him.

I express my deep sense of gratitude to professor A.H. Siddiqui, Dean Faculty of Science and Prof. Z.U. Ahmad Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh for not only providing me necessary facilities but also helping hands in meeting the difficult situations during the work.

I should also be grateful to all senior members of the Department of Mathematics, specially Dr. M.A. Qadri and Prof. Izhar Husain for their encourageous support and useful suggestions during the period of my research work.

I am grateful to the honourable Vice Chancellor for granting me the study leave. I am also thankful to the manager and the principal of my school for the required recommendations.

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Shabbir Khan

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LIST OF SYMBOLS

$\mathcal{B}_{CK}$	: Category of BCK-algebras
$\mathcal{B}_{CK(1)}$	: Category of bounded BCK-algebras
$\mathcal{B}_{CK(\wedge)}$	: Category of commutative BCK-algebras
$\mathcal{B}_{CK(1,\wedge)}$	: Category of bounded commutative BCK-algebras
$\mathcal{B}_{CK(i+)}$	: Category of positive implicative BCK-algebras
$\mathcal{B}_{CK(i)}$	: Category of implicative BCK-algebras
$\mathcal{B}_{CK(s)}$	: Category of semi-simple BCk-algebras
$\mathcal{B}_{CK(c)}$	: Category of complete BCK-algebras
$\mathcal{B}_{CK(pc)}$	: Category of pre-complete BCk-algebras
$\mathcal{B}_{CK(r)}$	: Category of BCK-algebras with regular morphisms
$\mathcal{B}_{CI}$	: Category of BCI-algebras
$\mathcal{B}_{CI(a)}$	: Category of associative BCI-algebras
$\mathcal{P}_{s}$	: Category of p-semi simple BCI-algebras
$\mathcal{B}_{(r)}$	: Category of BCK-algebras and retractions
$\mathcal{B}_{(cr)}$	: Category of BCK-algebras and co-retractions
I(X)	: Catdgory of involutions on X
P(X)	: Category of prime ideals of X
F(X)	: Category of filters of X
$\mathrm{Hom}_{\mathcal{B}_{CK}}(A,B)$	: Set of moprhisms from A to B in $\mathcal{B}_{CK}$
$\mathcal{S}$	: Category of sets
$\mathcal{A}b$	: Category of abelian groups
$\mathcal{L}at$	: Category of lattices
F,G,T,C,	: Functors
$\prod_{i\in I} Ai$	: Categorial product of the family $\{Ai\}_{i\in I}$

Ker(f)	:	Kernel of the morphism f
Cok(f)	:	Co-kernal of the morphism f
Im(f)	:	Image of the morphism f
Co-im(f)	:	Co-image of the morphism f.
$\cup$	:	Union
$\cap$	:	Intersection
$\subseteq$	:	Contained in
===>	:	Implies

CONTENTS

CHAPTER I

PRELIMINARY CONCEPTS ON BCI & BCK-ALGEBRAS

1.1 Introduction: The object of this chapter is to introduce basic notions and results of BCK & (BCI)-algebras on the basis of which the subsequent chapters of the thesis have been developed. The intention of the author is to cite all the results without claiming any originality, therefore, it has become imperative to state all such results without proofs.

Section 1.2 deals with the preliminary definitions and characteristics of different types of BCK (BCI)-algebras. In Section 1.3 some well known examples of BCK and BCI-algebras are given. Ideals and filters in BCK-algebras have been discussed in Section 1.4. The concept of homomorphisms and exact sequences in BCK & (BCI)-algebras has been given in Section 1.5. Some well known results on BCK and BCI-algebras have been cited in Section 1.6.

As no text book is available on the subject, the material of this chapter has been selected from the original papers available on the subject such as those of K. Iseki [18], Iseki and Tanaka [20], [19], Deeba and Thaheem [10], Young Bae Jun [22], Tiande, Lei [24].

1.2 Some Definitions and Characterizations:

In 1966, Imai, Y. and Iseki, K. drew an algebraic system, BCK-algebra from operation on sets and propositional calculi, cf. [15]. In 1980, Iseki defined a more general algebraic system called BCI-algebra, cf. [16]. Here different types of algebras with their characteristics have been pointed out.

Definition 1.1 A *BCI-algebra* $X = \langle X, *, \leq, 0 \rangle$ of the type $\langle 2,0 \rangle$ is an algebra satisfying the following axioms for all $x, y, z \in X$:

(A 1) $(x*y)*(x*z) \leq z*y$,

(A 2) $x*(x*y) \leq y$,

(A 3) $x \leq x$,

(A 4) $x \leq y$ and $y \leq x \implies x = y$,

(A 5) $x \leq 0 \implies x = 0$,

(A 6) $x \leq y$ if and only if $x*y = 0$.

If in BCI-algebra X, we replace axiom (A5) by

(A 5)' $0 \leq x$ for all $x \in X$,

then X is called a *BCK-algebra*.

Remark 1.1: Every BCK-algebra is a BCI-algebra.

Definition 1.2. Let X be a BCI (BCK)-algebra. A subset S of X is called *sub-algebra of X* if $0 \in S$ and for any $x, y \in S$, $x*y \in S$.

Definition 1.3. If there is an element 1 of a BCI (BCK)-algebra X satisfying $x \leq 1$ for all $x \in X$, then 1 is called the *Unit* of X. A BCI (BCK)-algebra X with unit is called to be *bounded*. In bounded algebra we denote $1*x$ by N_x.

Definition 1.4. A BCK-Algebra X is said to be *commutative* if $x \wedge y = y \wedge x$ for all $x,y \in X$, where $x \wedge y$ is defined by $y*(y*x)$.

Remark 1.2. In a bounded commutative BCK-algebra X we define,

(B 1) $\qquad x \vee y = N(N_x \wedge N_y)$.

Proposition 1.1. Any *bounded commutative* BCK-algebra is a *lattice*.

Remark 1.3. Since in a bounded commutative BCK-algebra X we have $x \wedge N_x = 0$ and $x \vee N_x = 1$ for all $x \in X$. It follows that each $x \in X$ has a complement N_x which is unique.

Proposition 1.2. Any bounded commutative BCI(BCK)-algebra is a uni-complemented lattice.

Definition 1.5. If an element $x \in X$ satisfies $NN_x = x$, then x is called an *involution*.

Proposition 1.3. The set I(X) of all involutions of a bounded BCI(BCK)-algebra X is a bounded BCI(BCK)-algebra.

Definition 1.6. If in a BCI-algebra X, the equality

(B 2) $\qquad (x*y)*z = x*(y*z)$

holds for all $x, y, z \in X$, then it is called to be an *associative* BCI-algebra.

Remark 1.4. An associative BCK-algebra is trivial.

Proposition 1.4. A BCI-algebra satisfying (B 2) is a *group* in which every element is an involution.

Definition 1.7. Let X be a BCI-algebra. The set $B = \{x \mid o \leq x\}$ is called *BCK-part of X* or *p-radical* of X. BCK-part of X is a *sub-algebra* of X.

Definition 1.8. If BCK-part of a BCI-algebra X is trivial then X is said to be *p-semi-simple* BCI-algebra.

Definition 1.9. Let X be a BCI-algebra, then $x, y \in X$ are called *comparable* iff $x \leq y$ or $y \leq x$.

Definition 1.10. Let X be a BCI-algebra. Choose an element $x_o \in X$ such that there does not exist $y \in X$ with $y \leq x_o$, then a subset $A(x_o) = \{x \in X \mid x_o \leq x\}$
is said to be *comparable* if each pair $x, y \in A(x_o)$ is *comparable*. We call $A(x_o)$ as *comparable branch*.

If $A(x_o)$ is a singleton set then it is said to be *uniary comparable*.

Definition 1.11. A proper BCI-algebra X is called S_4-*algebra* if each $A(x_o)$ is *uniary comparable* in X.

Definition 1.12. A BCI-algebra X is said to be a *medial BCI-algebra* if it satisfies,

(B 3) $(x*y)*(z*u) = (x*z)*(y*u)$ for all $x,y,z,u \in X$.

Remark 1.5. In a medial BCI-algebra X, the identity

(B 4) $x*(x*y) = y$ holds.

Remark 1.6. In a medial BCI-algebra X, Hom(X), the set of all homomorphisms from X to X is a medial BCI-algebra. It is not true in BCI-algebra.

Definition 1.13. Let X be a BCI-algebra then a subset

$G = \{ x \in X \mid o*x = x\}$ of X is called to be *G-part of X*.

Definition 1.14. A BCK-algebra X is said to be *positive implicative* if the equality

(B 5) $(x*z)*(y*z) = (x*y)*z$

holds for all $x,y,z \in X$. [20].

Proposition 1.5. A BCK-algebra X is *positive implicative* if and only if the equality

(B 6) $x*y = (x*y)*y$ holds.

Definition 1.15 If in a BCK-algebra X, the equality

(B 7) $x*(y*x) = x$

holds for all $x,y \in X$, then X is called to be *implicative*.

Proposition 1.6. Any implicative BCK-algebra is commutative and positive implicative.

Definition 1.16. A BCK-algebra X is said to be *with Condition (S)* if for any fixed elements $y,z \in X$ the set $A_{(y,z)} = \{x \in X \mid x*y \leq z\}$ has the greatest element which is denoted by $y \circ z$.

Definition 1.17. A BCK-algebra which is closed under supremum (S) and infimum (S) for every subset S of X is called a *complete BCK-algebra*.

Definition 1.18. A BCK-algebra X is said to be *pre-complete* if for every pair of elements $x,y \in X$ the set $\{x,y\}$ has supremum and infimum.

Remark 1.7. Every complete algebra is pre-complete.

Remark 1.8. To follow the standard terminology, we shall denote Sup $(\{x,y\})$ by $x \vee y$ and Inf $(\{x,y\})$ by $x \wedge y$.

1.3 Examples:

E_1: Let A be an arbitrary non-empty set and X be the set of all real valued functions defined on A. For any $f,g \in X$ we define $f*g$ by

$$(f*g)(x) = \begin{cases} 0, & \text{if } f(x) \leq g(x), \\ f(x)-g(x), & \text{if } g(x) < f(x). \end{cases}$$

By the definition of $*$, X is a commutative BCK-algebra.

E_2: Let Y be the set of all non-negative integer valued functions on an arbitrary non-empty set A. We define $f*g$ as follows:

$$(F*g)(x) = \begin{cases} 0, & \text{if } f(x) \le g(x), \\ 1, & \text{if } g(x) < f(x) \text{ and } g(x) \ne 0 \\ f(x), & \text{if } g(x) < f(x) \text{ and } g(x) = 0. \end{cases}$$

Y is a non-commutative BCK-algebra.

E_3: Consider the set $X = \{0, 1, \ldots, w\}$. Define $x*y$ as follows:

$$x*y = \begin{cases} 0, & \text{if } x \le y \le w, \\ 1, & \text{if } y < x < w \text{ and } y \ne 0, \\ w, & \text{if } y < x = w \text{ and } y \ne 0, \\ x, & \text{if } y < x \le w \text{ and } y = 0. \end{cases}$$

X is a bounded BCK-algebra with *unit* w.

E_4: Let $X = \{0,a,b\}$. Define a binary operation $*$ by the following table:

*	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

X is a BCI-algebra which is not BCK-algebra.

E_5: Z the set of integers in which the binary operation is defined as $x*y = x-y$ is a BCI-algebra. Since $0-x = 0 \Rightarrow x = 0$, its BCK-part must be trivial. Hence Z is a p-semi-simple BCI-algebra.

E_6: Let X' be a partially ordered set which contains a least element c. Let 0 be an element which is not contained in X' and consider $X = X' \cup \{0\}$ with $c > 0$. Now X is a partially ordered set containing the least element 0.

We define binary operation * on X as follows:

$$x*y = \begin{cases} x, & \text{for } y = 0, \\ 0, & \text{for } x \leq y, \\ c, & \text{for the other cases.} \end{cases}$$

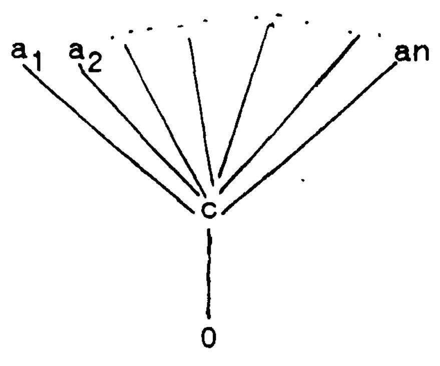
Then X is a BCK-algebra.

The following special cases may also be considered.

E_7: Let $X' = \{c\}$ then $X = \{0,c\}$ is a BCK-algebra with respect to the binary operation * as defined in example E_6. It is a positive implicative and commutative algebra. Further X be an algebra with condition (S).

E_8: Consider $X' = \{c,a\}$, then $X = X' \cup \{0\} = \{0,c,a\}$. In this case X is not a positive implicative BCK-algebra, because $x*y = (x*y)*y$ does not hold for $x = a$, $y = c$. But X is a *commutative*. Moreover, X is an algebra with condition (S), because $0 \leq x*y \leq c$ holds for all x and y. Hence,if $z=0$, $y \circ z$ is a, since $z \geq c$.

E_9: Assume $X' = \{c,a_1,a_2 \ldots, a_n)$, where a_1, $a_2 \ldots, a_n$ are not comparable (see figure below). Then X is a BCK-algebra. X is *not positive implicative*, since



$x*y = (x*y)*y$ does not hold for $x = a_i$, $y = c$. On the other hand X is commutative, since $x*(x*y) = y*(Y*x)$. For example, take $x = a_i$, $y = a_j$ $(i \neq j)$ then $x*\,(x*y) = y*\,(y*x) = c$.

E_{10}: Consider $M = \{0,1,2,....\}$ the set of all non-negative integers. For any two elements $x,y \in M$, define binary operation $*$ as follows:

$$x*y = \begin{cases} 0 & , \text{ if } x \leq y \\ x-y & , \text{ if } y < x. \end{cases}$$

Then M is a BCK-algebra. Moreover, M is a *commutative* BCK-algebra.

E_{11}: Let $M = \{0,1,2,....\}$ with the natural order. Then we define,

$$x*y = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{if } y < x \text{ and } y \neq 0 \\ x & \text{if } y < x \text{ and } y = 0 \end{cases}$$

Under the definition of $*$, M is a BCK-algebra. But M is not commutative BCK-algebra, i.e., $x \wedge y \neq y \wedge x$ for some $x,y \in M$.

E_{12}: Let M be a set with least element 0 such that every pair of non-zero distinct elements is incomparable. Then $M = \langle M,*,0\rangle$ is a BCK-algebra if and only if for any $x,y \in M$, we have

$$x*y = \begin{cases} x & \text{if not } (x \leq y) \\ 0 & \text{if } x \leq y. \end{cases}$$

M is an *I-algebra (Implicative algebra)*. Moreover, M is *commutative*.

E_{13}: Let M be a partially ordered set with the least element 0. $x*y = 0$ if $x \leq y$ and $x*y = x$ if not $(x \leq y)$. Then M is a BCK-algebra. Hence any partially ordered set can be regarded as a BCK-algebra.

1.4 Ideals and Filters in BCI and BCK-Algebras:

Definition 1.19: A non-empty subset A of an algebra X is called an *ideal* of X if

(i) $0 \in A$

(ii) $y, x*y \in A$ imply $x \in A$.

Definition 1.20: A non-empty subset A of a BCK-algebra X is called an *implicative ideal* of X if

(i) $0 \in A$

(ii) $y*z, (x*y)*z \in A$ imply $x*z \in A$.

Remark 1.9: Every implicative ideal is an ideal.

Proposition 1.7: An algebra X is implicative if and only if every *ideal* of X is an *implicative ideal*.

Definition 1.21: A proper ideal A of a BCK-algebra X is said to be *maximal* if it is not properly contained in any proper ideal of X.

Definition 1.22: Let A be a nonempty subset of a BCK-algebra X, then the set B of elements $x \in X$ such that there are elements $a_0, a_1, \ldots a_n \in A$, satisfying the equation.

(B 8) $$(\ldots((x*a_0)*a_1)\ldots)*a_n) = 0$$

is a minimal ideal containing A. The ideal B is called the ideal generated by A. If A is finite then we say that B is finitely generated.

Definition 1.23: Let c be an element of an implicative BCK-algebra X. Then the set: $\{x \in X \mid x \leq c\}$ is an ideal of X, which will be called a *principal ideal* generated by c.

Definition 1.24. An ideal A of a BCK-algebra X is called *irreducible ideal* if $A = B \cap C$ implies either $A = B$ or $A=C$ for ideals B and C.

Definition 1.25. An ideal A in a commutative BCK-algebra X is called *prime* if for any a,b, $a \wedge b \in A$ implies $a \in A$ or $b \in A$.

Definition 1.26. Let X be a bounded BCK-algebra then the *radical* of X is defined as the set: $\bigcap\{M \mid M$ is a maximal ideal of $X\}$. We denote radical of X by R(X). If $R(X) = 0$, then X is said to be *semi-simple* BCK-algebra.

Proposition 1.8. If X is a bounded positive implicative BCK-algebra, then

$$R(X) = \{x \in X \mid 1*x = 1\}.$$

Definition 1.27. Let A be an ideal of a BCI-algebra X and for every $x \in A$, $0 \leq x$, then A is said to be a *positive ideal* of X or briefly *p-ideal* of X.

Remark 1.10. BCK-part of BCI-algebra X contains all p-ideals of X. So it is a maximal p-ideal of X.

Definition 1.28. If a BCI-algebra X coincides with P(X) then it is called to be *p-radical algebra*.

Definition 1.29. A bounded BCK-algebra is said to have *maximal condition* if it has a unique non-trivial ideal.

Proposition 1.9. Any sub-algebra of a p-semi-simple BCI-algebra is an ideal.

Definition 1.30. Let A be an ideal of BCK-algebra X. For any, $x, y \in X$ we define

(B 9) $x \sim y \iff x*y$ and $y*x \in A$.

The relation ~ is an equivalence relation on X. We denote by C_x the equivalence class containing x and by X/A the set of all equivalence classes.

A binary operation * on X/A is defined by

(B 10) $C_x * C_y = C_{x*y}$.

Under the binary operation * defined above X/A is a BCK-algebra called the *quotient algebra of X by A.*

Definition 1.31. A non-empty subset F of a BCK-algebra X is said to be a *filter* of X if

(F.1) $x \in F$ and $x \leq y$ imply that $y \in F$

(F.2) $x \in F$ and $y \in F$ imply that glb $\{x,y\} \in F$.

Remark 1.11. A proper filter does not contain 0.

Remark 1.12. If X is a bounded commutative BCK-algebra and F is a filter of X, then $1 \in F$.

Proposition 1.10. A non-empty subset F of a BCK-algebra X is a filter of X if and only if

(B 11) $x,y \in F \Longleftrightarrow \text{glb}\{x,y\} \in F$.

Definition 1.32. A non-empty subset of a BCK-algebra X is said to be $\wedge$ *-closed system* if for every pair of elements $x,y \in S$ implies glb $\{x,y\} \in S$.

Theorem 1.1. A non-empty subset F of a commutative BCK-algebra X is a filter of X iff

(B 12) $x,y \in F \Longleftrightarrow x \wedge Y \in F$

Proposition 1.11: If P is an ideal of a commutative BCK-algebra X, then X-P is a *filter* of X iff P is a *prime ideal*.

Proposition 1.12: The union and intersection of filters is a filter.

Definition 1.33: A proper filter U of a BCK-algebra X is said to be an *Ultra-filter* of X if for every filter F of X with $U \subset F$, it follows that U = F or F = X.

Proposition 1.13: Let X be a bounfed implicative BCK-algebra. Then a non-empty subset F of X is an *Ultra filter* if and only if X-F is a prime ideal.

Proposition 1.14. Let X be a bounded implicative BCK-algebra and F is a proper filter of X. Then $x \in F$ if and only if $1*x \in X-F$.

Definition 1.34. Let X be a bounded implicative BCK-algebra and F a *filter* of X. Then for $x,y \in F$ we say $x \backsim y$, if

$$1*(x*y) \in F,\ 1*(y*x) \in F.$$

Proposition 1.15. Let X be a bounded implicative BCK-algebra and $x \in X$, then there is no *proper filter* of X containing x and N_x simultaneously.

Proposition 1.16. If F_1 and F_2 be the filters of a *bounded implicative* BCK-algebra X. Then the set $F_1 \vee F_2 = \{a \vee b \mid a \in F_1,\ b \in F_2\}$ is a filter of X.

1.5. Homomorphisms and exact sequences:

Definition 1.35. Let X and Y be BCK (BCI)-algebras. A mapping $f: X \longrightarrow Y$ is called a *BCK(BCI)-homomorphism* if for all $x,y \in X$,

$$f(x*y) = f(x)*f(y).$$

Proposition 1.17: For a BCK-homomorphism $f: X \longrightarrow Y$.

$$f(o) = 0$$

Proposition 1.18: Any BCK-homomorphism is isotone, i.e.,

$$x \leq y \implies f(x) \leq f(y).$$

Definition 1.36: Let $f: X \longrightarrow Y$ be a BCK-homomorphism then the set : $\{ x \in X \mid f(x) = 0\}$ is called *Kernel of f* and is denoted by *Ker (f)*.

Proposition 1.19: Let $f: X \longrightarrow Y$ be a homomorphism then Ker(f) is an ideal.

Definition 1.37.. A homomorphism $f: X \longrightarrow Y$ is said to be a *monomorphism* if and only if it is injective. f is said to be an *epimorphism* if and only if it is *surjective.* A *bijective* homomorphism is called an *isomorphism.*

Proposition 1.20. Let f be an epimorphism from a BCK-algebra X onto a BCK-algebra Y. Then the quotient algebra X/Ker(f) is isomorphic to Y.

Proposition 1.21. Let X,Y,Z, be BCK-algebras, and let $h: X \longrightarrow Y$ be an epimorphism; $g: X \to Z$ be a homomorphism. If $Ker(h) \subseteq Ker(g)$, then there is a unique homomorphism $f: Y \longrightarrow Z$ satisfying $foh = g$.

Proposition 1.22. Let X,Y,Z be BCK-algebras, and let $g: X \to Z$ be a homomorphism;and let $h: Y \to Z$ be a monomorphism with $Im(g) \subseteq Im(h)$. Then there is a unique homomorphism $f: X \longrightarrow Y$ satisfying $g = hof$.

Definition 1.38: Let X,Y be BCK-algebras. A homomorphism $f: X \longrightarrow Y$ is called *trivial* if $f(x) = 0$ for all $x \in X$.

Proposition 1.23. The composition $h = gof$ of two homomorphisms $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ of BCK-algebras is *trivial* homomorphism if and only if $Im(f) \subseteq Ker(g)$.

Definition 1.39. Let $f:X \to Y$ and $g:Y \to Z$ be the BCK-homomorphisms. Then the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called *exact at* Y if $Ker(g) = Im(f)$.

If a sequence

$$X_0 \to X_1 \to X_2 \to \dots\dots \to X_n \to X_{n+1}$$

is *exact* at $X_1, X_2, \dots\dots, X_n$, then the sequence simply called an *exact sequence*.

Proposition 1.24. The sequence

(i) $0 \to X \xrightarrow{f} Y$ is exact iff f is injective,

(ii) $X \xrightarrow{f} Y \to 0$ is exact iff f is surjective.

Proposition 1.25. The sequence

(i) $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact iff $f = Ker(g)$,

(ii) $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff $g = Coker(f)$.

Proposition 1.26. The sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact if the statements

(a) $f = Ker(g)$ and g is epic

and (b) $g = \mathrm{Coker}(f)$ and f is mono.

are equivalent.

Proposition 1.27. In an arbitrary exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$$

of BCK-homomorphisms, the following statements are equivalent.

(i) f is an epimorphism

(ii) g is a trivial homomorphism,

(iii) h is a monomorphism.

1.6 Some Properties of BCK and BCI-Algebras:

Theorem 1.2: In a BCK-algebra X, we have the following for all $x,y,z \in X$;

(P 5.1) $x \leq Y \Longrightarrow x*z \leq y*z$ and $z*y \leq z*x$,

(P 5.2) $x \leq y,\ y \leq z \Longrightarrow x \leq z$,

(P 5.3) $(x*y)*z = (x*z)*y$,

(P 5.4) $x*y \leq z \Longrightarrow x*z \leq y$

(P 5.5) $x*y \leq x$

(P 5.6) $x*0 = x$.

Theorem 1.3. In a *bounded* BCK-algebra we have,

(P 5.7) $N1 = 0,\ N0 = 1$,

(P 5.8) $NN_x \leq x$,

(P 5.9) $N_x*N_y \leq y*x$,

(P 5.10) $y \leq x \Longrightarrow N_x \leq N_y$,

(P 5.11) $N_x*y = N_y*x$,

(P 5.12) $1 \wedge x = x$,

(P 5.13) $x \wedge 1 = NN_x$,

(P 5.14) $NNN_x = N_x$.

Theorem 1.4. In any bounded commutative algebra X, the following are true,

(P 5.15) $N1 = 0$, $N0 = 1$,

(P 5.16) $NN_x = x$,

(P 5.17) $N_x \vee N_y = N(x \wedge y)$,

(P 5.18) $N_x \wedge N_y = N(x \vee y)$.

Theorem 1.5. In a bounded commutative BCK-algebra the following conditions are equivalent;

(P5.19) $x = x*(y*x)$,

(P5.20) $x*y = (x*y)*y$,

(P5.21) $(x*z)*(y*z) = (x*y)*z$,

(P5.22) $x \wedge N_x = 0$,

(P5.23) $x \vee N_x = 1$,

(P5.24) $x = x*N_x$,

(P5.25) $N_x = N_x*x$,

(P5.26) $x*(x*N_y) \leq x*y$,

(P5.27) $x*(x*y) \leq x*N_y$,

(P5.28) $(x*(y*z))*(x*y) \leq x*N_z$.

Theorem 1.6. Let X be a bounded BCK-algebra with $NN_x = x$ for all $x \in X$. If $\{0\}$ is an implicative ideal then $R(X) = \{0\}$ i.e. X is *semi-simple*.

Theorem 1.7. If X is a bounded positive implicative and semi-simple BCK-algebra, then $NN_x = x$ for all $x \in X$.

Theorem 1.8. Let X be a bounded positive implicative BCK-algebra, then X is semi-simple iff $NN_x = x$ for all $x \in X$.

Theorem 1.9. If X be a bounded positive implicative BCK-algebra then the quotient algebra $X/_{R(X)}$ is semi-simple.

Theorem 1.10. In a BCI-Algebra X, the following properties are equivalent,

(P5.29) X is p-semi-simple,

(P5.30) $0*x = 0 \implies x = 0$,

(P5.31) $0*(0*x) = x$ for all $x \in X$,

(P5.32) $x*(0*y) = y*(0*x)$ for all $x,y \in X$.

Theorem 1.11. Any abelian group is a p-semi-simple BCI-algebra under the operation defined by $x*y = x-y$.

Theorem 1.12. If $\langle X, +, 0 \rangle$ be a group induced by p-semi-simple BCI-algebra $\langle X, *, 0 \rangle$. Then p-semi-simple algebra induced by $\langle X, +, 0 \rangle$ coincides with $\langle X, *, 0 \rangle$.

Proposition 1.28. In a medial BCI-algebra X, G-part of X is an ideal.

Proposition 1.29. If X is a medial BCI-algebra then Hom(X) is a BCI-algebra.

Theorem 1.13. In a BCI-algebra X, the following are equivalent,

(a) X is medial,

(b) X is p-semi-simple,

(c) X is S_4-algebra,

(d) X has the property;

$$x \in G,\ y \in X-G \Longrightarrow x*y,\ y*x \in X-G.$$

Chapter II

CATEGORIES OVER BCI AND BCK-ALGEBRAS

2.1 Introduction: The main object of this chapter is to introduce and discuss the categories of BCK and BCI-algebras. In Section 2.2 categories $\mathcal{B}_{CI}$ and $\mathcal{B}_{CK}$ are introduced. However, in Section 2.3 some special types of categories such as $\mathcal{P}_S$, $\mathcal{B}_{CI(m)}$, $\mathcal{B}_{CK(\wedge)}$, $\mathcal{B}_{CK(1)}$, $\mathcal{B}_{CK(i)}$, $\mathcal{B}_{CK(i_+)}$ etc. are constructed with the help of different types of BCI and BCK-algebras. In Section 2.4 we investigate the relation among different special categories. The Section 2.5 deals with the characterization of mono and epimorphisms in $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ categories. In Section 2.6 sub-object, quotient object, zero object and zero morphism in $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ categories are discussed. Lastly, in Section 2.7 categorial structures such as products, sums, kernels, co-kernels, images, co-images, intersection, pullbacks and pushouts are studied.

2.2 $\mathcal{B}_{CI}$ and $\mathcal{B}_{CK}$ Categories:

$\mathcal{B}_{CI}$ Category: Consider the classes of all BCI-algebras as the class of objects of the category and the class of all homomorphisms between BCI-algebras as the class of morphisms of the category. This collection forms a category. We call this category as the *Category of BCI-algebras* and denote it by $\mathcal{B}_{CI}$.

$\mathcal{B}_{CK}$ Category: If we consider the class of all BCK-algebras as the class of objects and the class of all homomorphisms between BCK-algebras as the class of morphisms, then a category is formed. We call this category as the *Category of BCK-algebras and denote it by* $\mathcal{B}_{CK}$.

2.3 Some Special Categories:

$\mathcal{P}_s$: We can construct a category by taking all p-semi -simple BCI-algebras as the class of objects of the category and the class of morphisms of the category being the all homomorphisms between them. We shall denote this category by $\mathcal{P}_s$.

$\mathcal{B}_{CI(m)}$: Assume the class of all medial BCI-algebras as the class of objects of the category and the class of all BCI-homomorphisms between them as the class of morphisms of the category. Category formed in this way is called the *Category of medial BCI-algebras.*

$\mathcal{B}_{CI(a)}$: *Category of associative BCI-algebras* be the category in which the class of objects be the class of all associative BCI-algebras and homomorphisms between them be the class of morphisms of the category.

$\mathcal{B}_{CK(\wedge)}$: The category $\mathcal{B}_{CK(\wedge)}$ can be constructed by taking all commutative BCK-algebras as the class of objects of the category and homomorphisms between them as the class of morphisms of the category. This category is known as the *category of commutative BCK-algebras.*

$\mathcal{B}_{CK(1)}$: The category $\mathcal{B}_{CK(1)}$ be the *category of bounded BCK-algebras* in which class of objects is the class of all bounded BCK-algebras and morphisms of the category be the all unit preserving BCK-homomorphisms between bounded BCK-algebras.

$\mathcal{B}_{CK(1,\wedge)}$: The *category* $\mathcal{B}_{CK(1,\wedge)}$ *of bounded commutative BCK-algebras* can be formed by taking the class of all bounded commutative BCK-algebras as the class of objects of the category and the class of all BCK-homomorphisms between them as the class of morphisms of the category.

$\mathcal{B}_{CK(i_+)}$: It is the *category of positive implicative BCK-algebras* where class of all positive implicative BCK-algebras is the class of objects of the category and homomorphisms between them form the class of morphisms of the category.

$\mathcal{B}_{CK(i)}$: Consider the category in which the class of all implicative BCK-algebras forms the class of objects of the category and homomorphisms between them as the class of morphisms of the category. We call this category as the *category of implicative BCK-algebras.*

$\mathcal{B}_{CK(1,i_+)}$: It is the *category of bounded positive implicative BCK-algebras* and homomorphisms between them.

$\mathcal{B}_{CK(c)}$: This category represents the *category of complete BCK-algebras* in which class of objects be the class of all complete BCK-algebras and class of morphisms be the class of all homomorphisms between them.

$\mathcal{B}_{CK(pc)}$: Consider the class of all *pre-complete BCK-algebras* as the class of objects of the category and the class of all homomorphisms between them as the class of morphisms of the category. Here the morphisms are $\wedge$ and $\vee$ preserving homomorphisms.

$\mathcal{B}_{CK(s)}$: We can construct a category by taking the class of all semi-simple BCK-algebras as the class of objects and the class of all homomorphisms between them as the class of morphisms of the category. We call this category as the category of semi-simple BCK-algebras.

$\mathcal{B}_{CK(r)}$: Consider the class of all BCK-algebras as the class of objects of the category and the class of all regular homomorphisms between them as the class of morphisms of the category.

2.4 Sub-Categories:

$\mathcal{B}_{CK}$ is a sub-category of $\mathcal{B}_{CI}$: Since every BCK-algebra is a BCI-algebra, therefore, the category $\mathcal{B}_{CK}$ is a sub-category of the category $\mathcal{B}_{CI}$.

Further,

$$\mathrm{Hom}_{\mathcal{B}_{CK}}(X,Y) = \mathrm{Hom}_{\mathcal{B}_{CI}}(X,Y) \ \forall\ X,Y \in \mathcal{B}_{CK}$$

Hence we observe that $\mathcal{B}_{CK}$ is a *full-sub-category* of $\mathcal{B}_{CI}$.

Categories $\mathcal{B}_{CI(m)}$ and $\mathcal{P}_s$ are full sub-Categories of $\mathcal{B}_{CI}$:

As every medial BCI-algebra and p-semi-simple algebra is a BCI-algebra. Thus the categories $\mathcal{B}_{CI(m)}$ and $\mathcal{P}_s$ both are the sub-categories of $\mathcal{B}_{CI}$. Moreover,

$$\mathrm{Hom}_{\mathcal{B}_{CI(m)}}(X,Y) = \mathrm{Hom}_{\mathcal{P}_s}(X,Y) = \mathrm{Hom}_{\mathcal{B}_{CI}}(X,Y)$$

for all $X,Y \in \mathcal{B}_{CI(m)}$ and $\mathcal{P}_s$ respectively $\Rightarrow \mathcal{B}_{CI(m)}$ and $\mathcal{P}_s$ are the *full sub-categories* of the category $\mathcal{B}_{CI}$.

Category $\mathcal{B}_{CK(r)}$ is a sub-category of $\mathcal{B}_{CK}$. Further, $\mathcal{B}_{CK(r)}$ is *not the full sub-category* of the category $\mathcal{B}_{CK}$.

$\mathcal{B}_{CK(1,\wedge)}$ is a sub-Category of $\mathcal{B}_{CK(\wedge)}$: Since each bounded commutative BCK-algebra is a commutative algebra. So the category $\mathcal{B}_{CK(1,\wedge)}$ is a sub-category of the category $\mathcal{B}_{CK(\wedge)}$.

$\mathcal{B}_{CK(1,\wedge)}$ is *not the full sub-category* of $\mathcal{B}_{CK(\wedge)}$.

$\mathcal{B}_{CK(1,\wedge)}$ is a sub-Category of $\mathcal{B}_{CK(1)}$: Every bounded commutative algebra is a bounded BCK-algebra $\Rightarrow \mathcal{B}_{CK(1,\wedge)}$ is the sub-category of the category $\mathcal{B}_{CK(1)}$. Also $\mathcal{B}_{CK(1,\wedge)}$ is a *full sub-category* of $\mathcal{B}_{CK(1)}$.

$\mathcal{B}_{CK(i)}$ is a full sub-Category of $\mathcal{B}_{CK(i_+)}$: As very implicative BCK-algebra is a positive implicative BCK-algebra, therefore $\mathcal{B}_{CK(i)}$ is a sub-category of the category $\mathcal{B}_{CK(i_+)}$. It is a *full sub-category* of $\mathcal{B}_{CK(i_+)}$.

$\mathcal{B}_{CK(i)}$ is a sub-Category of $\mathcal{B}_{CK(\wedge,i_+)}$: Since we know that every implicative algebra is commutative positive implicative. So the category $\mathcal{B}_{CK(i)}$ is a sub-category of $\mathcal{B}_{CK(\wedge, i_+)}$.

$\mathcal{B}_{CK(c)}$ is a full sub-Category of $\mathcal{B}_{CK(pc)}$: Every complete BCK-algebra is a pre-complete BCK-algebra, so the category $\mathcal{B}_{CK(c)}$ is a sub-category of the category $\mathcal{B}_{CK(pc)}$.

Further, it is trivial that

$$\mathrm{Hom}_{\mathcal{B}_{CK(c)}}(X,Y) = \mathrm{Hom}_{\mathcal{B}_{CK(pc)}}(X,Y) \ \forall\ X,Y \in \mathcal{B}_{CK(c)}$$

Hence category $\mathcal{B}_{CK(c)}$ is a *full sub-category* of $\mathcal{B}_{CK(pc)}$.

$\mathcal{B}_{CK(1,\wedge)}$ is a sub-Category of $\mathcal{L}_{at}$: By proposition 1.1 we have that every bounded commutative BCK-algebra is a lattice. Hence we observe that the category $\mathcal{B}_{CK(1,\wedge)}$ is a sub-category of the category $\mathcal{L}_{at}$ of lattices.

2.5 Criteria of mono and epimorphisms in $\mathcal{B}_{CI}$ and $\mathcal{B}_{CK}$ categories:

Theorem 2.1 For any morphism $f:X \to Y$ in $\mathcal{B}_{CI}$ ($\mathcal{B}_{CK}$), the following are equivalent:

(i) f is one-one,

(ii) f is left cancellable,

(iii) $Ker(f) = \{0\}$.

Proof (i) ==> (ii): Assume that $f:X \to Y$ is a one-one BCK-homomorphism and let h_1, h_2: $Z \to X$ be such that $foh_1 = foh_2$. Then for all $x \in X$; $f(h_1(x)) = f(h_2(x))$. Since f is mono ==> $h_1(x) = h_2(x) \quad \forall\, x \in X$ ==> $h_1 = h_2$. Thus we have f is left cancellable.

Further (ii) ==> (iii).

Suppose $f:X \to Y$ be left cancellable and $Ker(f) \neq \{0\}$ then there exists an element $x(\neq 0) \in Ker(f)$.

Now consider maps;

$i : Ker(f) \to X$ s.t. $i(x) = x \quad \forall\, x \in Ker(f)$

and $\quad j : Ker(f) \to X$ s.t. $j(x) = 0 \quad \forall\, x \in Ker(f)$

Since f is left cancellable, therefore, $foi = foj$ ==> $i = j$, which is a contradiction. Hence Ker(f) must be trivial i.e. $Ker(f) = \{0\}$.

(iii) ==> (i); consider $f:X \to Y$ be such that

$Ker(f) = \{0\}$ and for some h_1, h_2: $A \to X$,

$foh_1 = foh_2$. Then $f(h_1(a)) = f(h_2(a)) \;\forall\, a \in A$.

$\Longrightarrow f(h_1(a))*f(h_2(a)) = 0,$

$\Longrightarrow f(h_1(a)*h_2(a)) = 0,$ [f is a homomorphism]

$\Longrightarrow h_1(a)*h_2(a) \in Ker(f),$

$\Longrightarrow h_1(a)*h_2(a) = 0$

$\Longrightarrow h_1(a) \leq h_2(a)$ ... (i_1)

Further,

$foh_2 = foh_1 \Longrightarrow f(h_2(a)) = f(h_1(a)) \; \forall \; a \in A.$

$\Longrightarrow f(h_2(a)*f(h_1(a)) = 0$

$\Longrightarrow f(h_2(a)*h_1(a)) = 0$

$\Longrightarrow h_2(a)*h_1(a) \in Ker(f)$

$\Longrightarrow h_2(a)*h_1(a) = 0$

$\Longrightarrow h_2(a) \leq h_1(a)$ ... (i_2)

From (i_1) and (i_2) we get, $h_1 = h_2$.

Thus $h_1 = h_2$ whenever $foh_1 = foh_2 \Longrightarrow f$ is one-one. Hence the proof follows.

Corollary 2.1: A morphism in category $\mathcal{B}_{CI}$ ($\mathcal{B}_{CK}$) is mono iff it is one-one.

Definition 2.1: A morphism $f:X \to Y$ in $\mathcal{B}_{CI}$ ($\mathcal{B}_{CK}$) category is called *regular* if Im(f) is an ideal.

Theorem 2.2: For any morphism $f:X \to Y$ in $\mathcal{B}_{CI}$ ($\mathcal{B}_{CK}$) category the following are equivalent:

(i) f is onto,

(ii) f is regular and right cancellable.

Proof (i) ==> (ii) Let $f:X \to Y$ be an onto morphism in $\mathcal{B}_{CI}$, and for any BCI-algebra Z and homomorphisms $h_1, h_2: Y \to Z$, $h_1 o f = h_2 o f$. Since f is onto, so for any $y \in Y$, there exists $x \in X$ s.t. $y = f(x) \Rightarrow h_1(y) = h_1(f(x)) = h_2(f(x)) = h_2(y)$ $\Rightarrow h_1(y) = h_2(y)$ for all $y \in Y \Rightarrow h_1 = h_2$.

(ii) ==> (i): Suppose that $f:X \to Y$ is not onto, then there exists atleast an element $y \in Y$ s.t. $y \notin Im(f)$. Since f is a regular homomorphisms, therefore Im(f) is an ideal of Y. Thus $Y/_{Im(f)}$ is a BCI-algebra. We can, therefore, have natural homomorphisms O and $\eta: Y \to Y/_{Im(f)}$ with $Ker(\eta) = Im(f)$. We also have,

$$\eta(f(x)) = C_{f(x)} = C_o = Im(f)$$

and $O\, f(x) = C_o = Im(f)$

Thus we get $\eta o f = O o f \Rightarrow \eta = O$, a contradiction, since $\eta(y) = C_y \neq Im(f)$. Hence f must be an onto morphism.

Corollary 2.2: In category $\mathcal{B}_{CI(r)}$ $(\mathcal{B}_{CK(r)})$ a morphism $f:X \to Y$ is an epimorphism iff it is onto.

2.6 Sub, Quotient and Zero Objects in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) Category:

Let X be a BCK(BCI)-algebra, A sub-algebra A of X together with an inclusion morphism $u:A \to X$ is called the sub-object of X in $\mathcal{B}_{CK}$. In particular if A is an ideal of X then it is called an *ideal sub-object* of X.

Remark 2.1 In category $\mathcal{B}_{CK}$ every ideal sub-object is also a sub-object. But it is not true in category $\mathcal{B}_{CI}$.

Remark 2.2 The categories $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ are well powered. Since for any algebra X the collection of all sub-objects of X is a subset of the power set of X.

Let A be an ideal of X. The quotient algebra $X/_A$ together with natural projection $p:X \to X/_A$ is called a *quotient object* of X.

Remark 2.3 The categories $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ are co-well powered. Since for any algebra X the collection of all quotient objects of X is equivalent to a subset (set of all ideals of X) of the power set of X.

The set Z = {0} trivially forms a BCK-algebra i.e. $Z \in \mathcal{B}_{CK}$.

For any other object $X \in \mathcal{B}_{CK}$, there is only one morphism Z -> X that corresponds 0 to 0.

Further, there xists one and only one morphism X -> Z; x -> 0 for all $x \in X$. Thus we observe that Z = {0} is the *initial* and *terminal* object in $\mathcal{B}_{CK}$. Moreover, Z is the *zero object* of the category $\mathcal{B}_{CK}$. Thus we have,

Proposition 2.1 The category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has zero object.

Definition 2.2 For any pair of objects X, Y in $\mathcal{B}_{CK}$, there exists a morphism f:X -> Y s.t. f(x) = 0 for all $x \in X$ where $X \to Y = X \to Z \to Y$ with Z as the zero object in $\mathcal{B}_{CK}$. We call the map f as zero-homomorphism and denote it by 0.

Proposition 2.2 In category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) a morphism $f : X \to Y$ is a zero morphism iff $\ker(f) = X$.

2.7 Some Categorial Structures in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) Category

Product in $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ Categories:

Let $\{M_i\}_{i \in I}$ be a family of BCK-algerbas. Consider the set of all functions $f: I \to \cup M_i$ s.t. $f(i) \in M_i$. Denote this set by $\prod_{i \in I} M_i$

For any $f, g \in \prod_{i \in I} M_i$, define $f*g: I \to \cup M_i$ as

$$(f*g)(i) = f(i)*g(i) \text{ in } M_i$$

and $\quad 0: I \to \cup M_i$ s.t. $0(i) = 0$ in M_i.

Under the operation $*$ defined as above, $\prod_{i \in I} M_i$ forms a BCK-algebra.

Projections in $\prod_{i \in I} M_i$ may be defined as follows:

For any $i \in I$, we have $p_i: \prod M_i \to M_i$ s.t.

$$p_i(f) = f(i) \text{ for all } f \in \prod M_i$$

p_i is a BCK-homomorphism

For any $f, g \in \prod M_i$ we have,

$$\begin{aligned} p_i(f*g) &= (f*g)(i) \\ &= f(i)*g(i) \\ &= p_i(f)*p_i(g) \end{aligned}$$

$\Longrightarrow p_i(f*g) = p_i(f)*p_i(g)$

$\Longrightarrow p_i$ is a BCK-homomorphism i.e. p_i is a morphism in $\mathcal{B}_{CK}$.

Proposition 2.3: The object $\prod M_i$ together with the projections $p_i: \prod M_i \longrightarrow M_i$ is the product of the family $\{M_i\}_{i \in I}$ in $\mathcal{B}_{CK}$.

Proof: Consider a family $\{X \xrightarrow{\eta_i} M_i\}$ of morphisms in $\mathcal{B}_{CK}$. Define a function $\eta: X \longrightarrow \prod M_i$ s.t. for all $x \in X$,

$$\eta_{(x)}: I \longrightarrow \cup M_i,$$

$$(\eta_{(x)})(i) = \eta_i\ (x) \quad \text{for all } i \in I$$

Further, for any $x_1, x_2 \in X$ we have,

$$\begin{aligned}(\eta_{(x1*x2)})\ (i) &= \eta_i\ (x1*x2)\\ &= \eta_i\ (x1) * \eta_i\ (x2)\\ &= (\eta_{(x1)})\ (i) * (\eta_{(x2)})\ (i)\end{aligned}$$

$\Longrightarrow (\eta_{(x1*x2)})\ (i) = (\eta_{(x1)}(i) * (\eta_{(x2)})\ (i)$ for all $i \in I$

$\Longrightarrow \eta$ is a morphism in $\mathcal{B}_{CK}$

Now, for any $i \in I$ and $x \in X$ we get

$$\begin{aligned}(p_i o \eta)\ (x) &= p_i(\eta_{(x)})\\ &= (\eta_{(x)})\ (i)\\ &= \eta_i\ (x).\end{aligned}$$

i.e. $(p_i o\eta)(x) = \eta_i\ (x)$ for all $i \in I$ and $x \in X$

$\Longrightarrow p_i o\eta = \eta_i$ for all $i \in I$.

Finally to show that η is unique

Let there exists an other morphism $j: X \longrightarrow \prod M_i$ s.t. $p_i oj = \eta_i$. Then,

$$\begin{aligned}(p_i oj)(x) &= p_i(j(x))\\ &= (j(x))\ (i) = n_i\ (x) = (\eta_{(x)})\ (i)\end{aligned}$$

$\Longrightarrow (j(x))(i) = (\eta_{(x)})(i)$ for all $i \in I$

$\Longrightarrow j(x) = \eta_{(x)}$ for all $x \in X$

$\Longrightarrow j = \eta$

Hence, $\prod M_{i \, i \in I}$ is the product of the family $\{M_i\}_{i \in I}$ in $\mathcal{B}_{CK}$.

Corollary 2.3: Category $\mathcal{B}_{CI}$ has products.

Similarly we can show that category $\mathcal{B}_{CI}$ has products.

Sum in $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ Categories:

The categorial sum in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) category can be constructed by using free BCK-structures. Jia in [21] has already proved the existence and uniqueness of free BCK-algebras and categorial sum. We brief his work as follows:

Definition 2.3: Let X be a non-empty set. A free BCK-algebra T with a map $f: X \longrightarrow T$ s.t. for any BCK-algebra A and map $g: X \longrightarrow A$, there exists a unique homomorphism $\eta: T \longrightarrow A$ s.t. $\eta$of. = g.

Theorem 2.3: For any non-empty set X, there exists a unique free BCK-algebra generated by X upto isomorphism.

Assume the family $\{Ai\}_{i \in I}$ of BCK-algebras and Si be the set of generators such that Ai is free on Si (in case Ai is not free we take Si = Ai)

Consider the disjoint union ΣSi of $\{Si\}_{i \in I}$ and free algebra F (Σ Si). By the definition of free algebra for each $i \in I$, there exists a unique monomorphism $Ui: Ai \longrightarrow F(\Sigma Si)$.

The algebra F ($\leq$ Si) together with the injections Ui forms a categorial sum of $\mathcal{B}_{CK}$.

Proposition 2.4: The category $\mathcal{B}_{CK}$ has sums.

Kernels, Co-kernels, Images and Co-images in $\mathcal{B}_{CK}$

Proposition 2.5 Let $f:X \longrightarrow Y$ be a BCK-homomorphism.

The set $K = \{x \in X \mid f(x) = 0\}$ is an ideal.

Proposition 2.6 The sub-object K together with inclusion map $U:K \longrightarrow X$ is the *categorial kernel* of the homomorphism $f:x \longrightarrow Y$ in the category $\mathcal{B}_{CK}$.

Theorem 2.4 The category $\mathcal{B}_{CK}$ has kernels.

It is trivial that the category $\mathcal{B}_{CI}$ also has kernels.

Definitions 2.4 Let $f:X \longrightarrow Y$ be a homomorphism in $\mathcal{B}_{CK}$, then the set defined as $Im(f) = \{y \mid \exists\, x \in X;\ f(x)=y\}$ is called the image of f.

Proposition 2.7 If $f:X \longrightarrow Y$ be a BCK-homomorphism then Im(f) is a sub-algebra of Y.

Proof Let $y_1, y_2 \in Im(f)$, then $\exists\, x_1, x_2 \in X$ s.t. $f(x_1) = y_1$ and $f(x_2) = y_2$.

Further, $y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$

$\Longrightarrow y_1 * y_2 = f(x_1 * x_2)$ for all $y_1, y_2 \in Im(f)$

$\Longrightarrow y_1 * y_2 \in Im(f)$.

Since $f(0) = 0 \Longrightarrow 0 \in Im(f)$. Hence $Im(f)$ is a sub-algebra of Y.

Remark 2.4 $Im(f)$ need not be an ideal in general.

For the morphism $f:X \longrightarrow Y$ we have two morphisms; $f':X \dashrightarrow Im(f)$ s.t. $f'(x) = f(x)$ for all $x \in X$ which is an epimorphism (surjective) and morphism $U: Im(f) \dashrightarrow Y$, the inclusion map which is a monomorphism. It is clear that $f = uof'$ and $U: Im(f) \dashrightarrow Y$ is the smallest sub-object of Y through which f factored. Thus we have

Proposition 2.8 The category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has images.

Since $ker(f)$ is an ideal of X, we can form the quotient algebra $X/_{ker(f)}$. There are two morphisms related to f; $p:x \longrightarrow X/_{ker(f)}$ the natural projection which is an epimorphism and morphism $f'': Y/_{ker(f)} \longrightarrow Y$ defined as $f''(c_x) = f(x)$ which is a monomorphism. Further, we get a factorization $f = f''op$ where $p: X \longrightarrow X/_{ker(f)}$ be the smallest quotient object of X related to such factorization. Hence we have,

Proposition 2.9: The category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has co-images.

Proposition 2.10: For any homomorphism $f:X \longrightarrow Y$ in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$), Co-im (f) $\cong$ Im(f).

For any homomorphism $f:X \longrightarrow Y$, we can construct the ideal $\langle Im(f) \rangle$ of Y and the quotient algebra $Y/_{\langle Im(f)\rangle}$.

Proposition 2.11 For the morphism $f:X \longrightarrow Y$ in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$), the projection morphism $Y \longrightarrow Y/_{\langle Im(f)\rangle}$ is the co-kernel of f.

Corollary 2.4 Category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has co-kernels.

Theorem 2.5 For any homomorphism $f:X \longrightarrow Y$ in $\mathcal{B}_{CK(r)}$, we have the following:

(i) ker.cok(f) = Im(f)

(ii) cok.ker(f) = Co-im(f).

Equalizers in $\mathcal{B}_{CK}$

Theorem 2.6 Category $\mathcal{B}_{CK}$ has equalizers.

Proof Let f_1, f_2: $X \longrightarrow Y$ be a pair of morphisms in $\mathcal{B}_{CK}$. Consider the subset $K = \{x | x \in X;\ f_1(x) = f_2(x)\}$

$k \neq \emptyset$ since $f_1(0) = 0 = f_2(0)$ i.e. $0 \in K$.

Assume that x_1, $x_2 \in K$, so $f_1(x_1) = f_2(x_1)$ and $f_1(x_2) = f_2(x_2)$. Clearly $f_1(x_1*x_2) = f_2(x_1*x_2) \Longrightarrow x_1*x_2 \in K$.

Thus K is a sub-algebra of X with the property that

$$K \xrightarrow{u} X \xrightarrow{f_1} Y = K \xrightarrow{u} X \xrightarrow{f_2} Y$$

Let $v:M \longrightarrow X$ be another morphism in $\mathcal{B}_{CK}$.

s.t. $f_1 o v = f_2 o v$ i.e. $(f_1 o v)(m) = (f_2 o v)(m)$ for all $m \in M$.

So $f_1(v(m)) = f_2(v(m))$ for all $m \in M$

Hence $v(m) \in K$ for all $m \in M$.

Now we can define η: $M \longrightarrow K$ s.t.

$\eta(m) = v(m)$ for all $m \in M$. Trivially η is a morphism in $\mathcal{B}_{CK}$ s.t. $u o \eta = v$.

Hence K together with a morphism $u : K \longrightarrow X$ is the equalizer of the pair of morphisms f_1 and f_2 in $\mathcal{B}_{CK}$.

Thus we have every pair of morphisms has equalizers in $\mathcal{B}_{CK}$.

Corollary 2.5 *Let $f : X \longrightarrow Y$ be any homomorphism in $\mathcal{B}_{CK}$.* The equalizer of the pair of morphisms $(f, 0)$ is the kernel of f.

Theorem 2.7 The category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has co-equalizers.

Proof Let f_1, f_2: $X \longrightarrow Y$ be a pair of morphisms in $\mathcal{B}_{CK}$ and θ be the minimum ideal congruence on Y such that $f_1(x) \equiv f_2(x)$ (θ) for all $x \in X$.

Now consider quotient algebra Y/θ and the canonical morphism $p : Y \longrightarrow Y/\theta$. By the definition of congruence relation θ, we get $pof_1 = pof_2$.

For any other morphism $q : Y \longrightarrow Z$ in $\mathcal{B}_{CK}$ such that $qof_1 = qof_2$, there is an associated ideal congruence relation R with

$$f_1(x) = f_2(x) \ (R) \text{ for all } x \in X.$$

By the smallest congruence property of θ, there exists a unique morphism η: $Y/\theta \longrightarrow Z$ such that the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \arrow[r, "q"] \arrow[dr, "p"'] & Z \\ & & Y/\theta \arrow[u, "\eta"'] \end{tikzcd}$$

is commutative. Hence $p: Y \longrightarrow Y/\theta$ is the co-equalizer of f_1, f_2 in $\mathcal{B}_{CK}$.

Corollary 2.6 Let $f: X \longrightarrow Y$ be a homomorphism in $\mathcal{B}_{CK}$. The co-equalizer of the pair of maps $(f,0)$ is the co-kernel of f.

Theorem 2.8 Category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has intersections.

Proof: Let $\{Mi\}_{i \in I}$ be a family of sub-objects of the BCK-algebra M. Consider the set

$$M' = \bigcap_{i \in I} Mi \quad ; \; M' \neq \emptyset \text{ as } 0 \in Mi \text{ for all } i \in I,$$

so $0 \in M'$. With respect to the operation defined in M, M' is a sub-algebra of M in the category $\mathcal{B}_{CK}$.

Let $f: N \longrightarrow M$ be a monomorphism in $\mathcal{B}_{CK}$ such that $N \xrightarrow{f} M = N \xrightarrow{f_i} M_i \xrightarrow{u_i} M$ for all $i \in I$, where u_i are inclusion morphisms. Since each ui is mono $\Longrightarrow f(N) \subseteq Mi$ for all $i \in I$.
Hence $f(N) \subseteq \bigcap Mi = M'$.

Now we can define $\bar{f}: N \longrightarrow M'$ s.t. $\bar{f}(x) = f(x)$ for all $x \in N$ and $N \xrightarrow{f} M = N \xrightarrow{\bar{f}} M' \xrightarrow{u} M$. This shows that M' together with $u: M' \longrightarrow M$ is the intersection of the family of sub-objects $\{ui: Mi \longrightarrow M\}$ in the category $\mathcal{B}_{CK}$. Hence the theorem follows:

Theorem 2.9 Category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has pullbacks.

Proof: Since category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has products and equalizers. Therefore, by theorem 3.9 of Blyth [3] it has pullbacks.

Dually, we have

Theorem 2.10 Category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) has pushouts.

CHAPTER III

SOME STRUCTURAL OBSERVATIONS ABOUT $\mathcal{B}_{CK}$ AND $\mathcal{B}_{CI}$ CATEGORIES

3.1 Introduction: This chapter is devoted to the study of some structural properties of different categories of BCK and BCI-algebras. In section 3.2 it is observed that the category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) is uniquely factorizable, it is neither normal nor co-normal category. In section 3.3 it has been checked that the categories $\mathcal{B}_{CK(r)}$ and $\mathcal{B}_{CI(r)}$ are balanced, binormal, uniquely factorizable and exact. The last section 3.4 deals with the study of the category $\mathcal{P}_s$ (the category of p-semi-simple BCI-algebras). It is a nice category which has zero object, kernels, co-kernels, products and sums. It is a balanced, uniquely factorizable, exact and abelian category.

3.2 $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ Categories.

In chapter II we have observed that categories $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ have zero object, products, sums, equalizers, co-equalizers, kernels, co-kernels, images, co-images, intersections, pullbacks and pushouts. With the help of theorem 6.2 and its dual of Blyth [3] we can conclude that the categories $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ are *complete* and *co-complete* both. In the following result we will show that these categories are uniquely factorizable.

Theorem 3.1 Category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) is uniquely factorizable.

Proof: Let $f: X \longrightarrow Y$ be a morphism in category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$). Assume that $\ker(f) = A$ and consider a natural projection $p: X \longrightarrow X/A$.

Define $q: X/A \longrightarrow Y$ such that $q(C_x) = f(x)\ \forall\ x \in X$. Further, $q(C_x) = 0 \Longrightarrow f(x) = 0 \Longrightarrow x \in \ker(f)$. Thus we have $C_x = 0 \Longrightarrow \ker(q) = 0 \Longrightarrow q$ is a monomorphism. So the morphism f can be factorized as

$$X \xrightarrow{f} Y = X \xrightarrow{p} X/A \xrightarrow{q} Y$$

i.e. $f = qop$. where q is a mono and p is an epimorphism.

Next to show that f is uniquely factorizable; Let there exist morphisms $\alpha: X \longrightarrow B$ and $\beta: B \longrightarrow Y$ with β mono and α as epimorphism such that

$$f = \beta o \alpha$$

Consider the diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "\alpha"'] & X/A \arrow[r, "q"] \arrow[d, dashed, "\eta"] & Y \\ & B \arrow[ur, "\beta"'] & \end{tikzcd}$$

Now,

$$\begin{aligned} Ker(f) &= Ker(\beta o \alpha) \\ &= Ker(\alpha) \qquad \text{(as } \beta \text{ is monomorphism)} \end{aligned}$$

Thus we have, $A = Ker(f) = Ker(\alpha) = Ker(p)$.

By proposition 1.21, there exists a unique $\eta: X/A \longrightarrow B$ such that $\eta op = \alpha \Longrightarrow \eta op$ is an epimorphism $\Longrightarrow \eta$ is *epimorphism*. ... (i)

Further, $f = \beta o \alpha$

$= \beta o(\eta o p)$

$= (\beta o \eta) o p$

and also $f = q o p$

$\Longrightarrow (\beta o \eta) o p = q o p$

$\Longrightarrow \beta o \eta = q$ (as p is epimorphism)

But q is *mono* ===> $\beta o \eta$ is *mono* ===> η is *monomorphism* ...(ii)
From (i) and (ii) we get η is an *isomorphism*. Hence the proof follows.

Theorem 3.2 Category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) is not normal.

Proof: Let $f: X \longrightarrow Y$ be a morphism in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$). The co-kernel of f is the morphism $\emptyset: Y \longrightarrow Y/\langle Im(f)\rangle$. Where $\langle Im(f)\rangle$ is the ideal generated by Im(f) in $\mathcal{B}_{CK}$ and the kernel of $\emptyset$ is the embedding $i: \langle Im(f)\rangle \longrightarrow Y$. But $\emptyset o f = 0$ ===> there exists a unique morphism $\eta: X \longrightarrow \langle(Im(f)\rangle$ such that $i o \eta = f$. In case f is a monomorphism and also a kernel, then η must be an isomorphism. But it is not, since $X \xrightarrow{\eta} \langle Im(f)\rangle = X \cong Im(f) \xrightarrow{j} \langle Im(f)\rangle$. Where j is an inclusion morphism in general. Hence f is not a kernel. Thus every *monomorphism in* $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) *need not be a kernel. This shows that the category* $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) *is not normal.*

Dually,

Theorem 3.3 Category $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) is not co-normal.

Proof: Let $f: X \longrightarrow Y$ be a morphism in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$). The morphism u:Ker(f) ⟶ X is the kernel of f in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$). The natural projection p: X ⟶ X/ker(f) is the co-kernel of u in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$). But fou = 0. By the definition of co-kernel there exists a unique morphism Ø: X/ker(f) ⟶ Y such that

$$\emptyset op = f$$

If f is an epimorphism and also a co-kernel, we observed that Ø must be an isomorphism, but it is not true in general since every epimorphism in $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) need not be surjective.

3.3 $\mathcal{B}_{CK(r)}$ and $\mathcal{B}_{CI(r)}$ Categories

Categories $\mathcal{B}_{CK(r)}$ and $\mathcal{B}_{CI(r)}$ are the sub-categories of the categories $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ respectively. Thus all the properties of $\mathcal{B}_{CK}$ ($\mathcal{B}_{CI}$) should be inherited in $\mathcal{B}_{CK(r)}$ ($\mathcal{B}_{CI(r)}$). Further by corollary 2.2 a morphism in $\mathcal{B}_{CK(r)}$ is epimorphism if and only if it is surjective. Hence we have,

Theorem 3.4 Category $\mathcal{B}_{CK(r)}$ ($\mathcal{B}_{CI(r)}$) is balanced.

Theorem 3.5 Category $\mathcal{B}_{CK(r)}$ ($\mathcal{B}_{CI(r)}$) is normal.

Proof: Let f:X ⟶ Y be a monomorphism in $\mathcal{B}_{CK(r)}$ ⟹ X ≅ Im(f) is an ideal of Y.

Consider the projection p:Y ⟶ Y/Im(f) where pof = 0. The inclusion morphism i: Im(f) ⟶ Y is the kernel of p in $\mathcal{B}_{CK(r)}$. By isomorphism property we get that

$$X \xrightarrow{f} Y = X \cong Im(f) \xrightarrow{i} Y$$

is also a kernel of p.

Theorem 3.6 Category $\mathcal{B}_{CK(r)}$ is co-normal.

Proof: Let f: X ⟶ Y be an epimorphism in $\mathcal{B}_{CK(r)}$ ===> f is surjective. Then kernel of f is the incluson map i: ker(f) ⟶ X i.e. foi = 0.

If g: X ⟶ Z be any other morphism such that goi = 0, then trivially ker(f) $\subseteq$ ker(g). Since f is surjective, by proposition 1.21 there exists a unique morphism η: Y ⟶ Z such that the diagram

$$\begin{tikzcd} Ker(f) \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, dashed, "\eta"] \\ & Z & \end{tikzcd}$$

is commutative i.e. $\eta$of = g. Hence f is a co-kernel of i.

Theorem 3.7 Category $\mathcal{B}_{CK(Y)}$ is binormal.

Corollary 3.1 Category $\mathcal{B}_{CK(r)}$ ($\mathcal{B}_{CI(r)}$) is uniquely factorizable.

Proof: Binormal category is uniquely factorizable [3].

Theorem 3.8 Category $\mathcal{B}_{CK(r)}$ is exact.

Proof: Since the category $\mathcal{B}_{CK(r)}$ has zero object, kernel and co-kernels, and every morphism f: X ——> Y in it can be decomposed as

$$X \xrightarrow{f} Y = X \xrightarrow{v} I \xrightarrow{u} Y$$

such that u mono and v epimorphism. Hence $_{CK(r)}$ is an exact category.

3.4 Category $\mathcal{P}_s$

The category $\mathcal{P}_s$ is a full sub-category of the category $\mathcal{B}_{CI}$, therefore all the properties of $\mathcal{B}_{CI}$ should be inherited in $\mathcal{P}_s$ i.e. it has zero object, kernel, co-kernel, products, sums, equalizers, co-equalizers, limits, co-limits, well powered and co-well powered etc. It is a nice algebraic category having the following individual properties.

Proposition 3.1 If f: X ——> Y be a BCI-homomorphism and Y is a p-semi simple BCI-algebra, then Im(f) is an ideal of Y.

Corollary 3.2 Every homomorphism with p-semi simple co-domain is regular.

Theorem 3.9 $\mathcal{P}_s$ is a regular category.

Proof: Follows from Cor. 3.2.

Theorem 3.10 Category $\mathcal{P}_s$ is balanced.

Proof: Follows from Theorems 2.1, 2.2 and 3.9.

Proposition 3.2: Category $\mathcal{P}_s$ has *zero object.*

Proof: Trivially the object {0} is a p-semi simple BCI-algebra ===> $\{0\} \in \mathcal{P}_s$.

Proposition 3.3: Category $\mathcal{P}_s$ has *kernels.*

Proof: Since the category $\mathcal{P}_s$ is a full sub-category of the category $\mathcal{B}_{CI}$, which has kernels. Further any sub-algebra of a p-semi simple algebra is a p-semi simple algebra ===> kernel of any morphism in $\mathcal{P}_s$ is p-semi simple ===> $\mathcal{P}_s$ has kernels.

Proposition 3.4 $\mathcal{P}_s$ has co-kernels.

Proof: Since $\mathcal{P}_s$ is a regular category. Therefore, for any homomorphism f: X ——> Y in $\mathcal{P}_s$, the morphism p: Y ——> Y/Im(f) is the co-kernel of f.

Proposition 3.5 Every monomorphism in $\mathcal{P}_s$ is a kernel.

Proof: Let u: K ——> X be a monomorphism in $\mathcal{P}_s$. Then by Theorem 2.1, u is injective ===> K $\cong$ Im(u) $\subseteq$ X, where Im(u) is an ideal of X (by prop 3.1).

Consider the projection morphism p: X ——> X/Im(u). The monomorphism u: K ——> X is the kernel of the morphism p. For any other homomorphism f:A ——> X in $\mathcal{P}_s$ with pof = 0, we have Im(f) $\subseteq$ ker (p) = Im(u) $\cong$ K. Now, by prop. 1.21 We can define a *unique* homomorphism η: A ——> K s.t. for any a $\in$ A.

$\eta(a) = b$ where $u(b) = f(a)$ and trivially $u \circ \eta = f$.

Proposition 3.6 Every epimorphism in $\mathcal{P}_s$ is a co-kernel.

Proof: Each epimorphism is a co-kernel of its kernel in $\mathcal{P}_s$.

Theorem 3.11: $\mathcal{P}_s$ is a binormal category.

Theorem 3.12: Category $\mathcal{P}_s$ in uniquely factorizable.

Let X, Y be BCK-algerbas. An operation * on the cartesian product X x Y of X, Y is defined as

$$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2,\ y_1 * y_2),$$

$$0 = (0,0)$$

and $(x_1, y_1) \leq (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$

The set X x Y with respect to the operation * (as defined above) and 0 = (0,0) is a BCI-algebra, called the product of X and Y.

Proposition 3.7: For any BCI-algebra X and Y,

$$R\,(X \times Y) = R\,(X) \times R\,(Y).$$

Proof: For any $(x,y) \in X \times Y$ we have,

$$0 \leq (x,y) \iff 0 \leq x \text{ and } 0 \leq y$$

Hence the proof follows.

Corollary 3.3: The product X x Y is p-semi-simple iff X and Y both are p-semi-simple.

Lemma 3.1: BCI-algebra X x Y together with the projections p_1, p_2: X x Y ⟶ Y defined by,

$$p_1\,(x,y) = x \text{ and } p_2(x,y) = y$$

is the categorial product of X and Y in $\mathcal{P}_s$.

Proof: Let $f_1: Z \longrightarrow X$ and $f_2 : Z \longrightarrow Y$ be any pair of morphisms in $\mathcal{P}_s$. We can define a function

$$\eta: Z \longrightarrow X \times Y \text{ s.t. } \eta(z) = (f_1(z), f_2(z)) \ \forall z \in Z$$

η is a BCI-homomorphism

$\eta(0) = (f_1(0), f_2(0) = (0,0)$ (since f_1, f_2 are BCI-homomorphisms)

Now, for any $z_1, z_2 \in Z$,

$$\begin{aligned}\eta(z_1*z_2) &= (f_1(z_1*z_2), f_2(z_1*z_2)) \\ &= (f_1(z_1)*f_1(z_2), f_2(z_1)*f_2(z_2)) \\ &= (f_1(z_1), f_2(z_1)) * (f_1(z_2), f_2(z_2)) \\ &= \eta(z_1) * \eta(z_2)\end{aligned}$$

$\Longrightarrow \eta(z_1*z_2) = n(z_1) * n(z_2) \ \forall z_1, z_2 \in Z$

$\Longrightarrow \eta$ is a $\mathcal{B}_{CI}$-homomorphism.

Trivially, $p_1 \circ \eta = f_1$ and $p_2 \circ \eta = f_2$

Hence $X \times Y$ forms the categorial product of X, Y.

Theorem 3.13: Category $\mathcal{P}_s$ has product for every pair of objects.

Theorem 3.14: Category $\mathcal{P}_s$ is an exact category.

Proof: Follows from Theorem 3.12 and propositions 3.2, 3.3, 3.4.

Lei Tiande and Xi, Chang chang in [25] have shown that every p-semi simple algebra forms an abelian group w.r. to the operation defined as

$$x + y = x * (0*y)$$

and every abelian group G forms a p-semi simple algebra with respect to the operation defined as

$$x * y = x - y$$

Lemma 3.2: For p-semi-simple algebras X and Y, the p-semi simple algebra X x Y together with the natural inclusions u_X: X ⟶ X x Y and u_y: Y ⟶ X x Y forms a categorial sum (co-product) of X and Y.

Proof: For any pair of morphisms f:X ⟶ Z, g: Y ⟶ Z where Z is also a p-semi simple BCI-algebra, we can define a morphism η: X x Y ⟶ Z s.t.

$$\eta(x,y) = f(x) + g(y) = f(x) * (0 * g(x))$$

η is a BCI-homomorphism:

For $(x_1,y_1),\ (x_2,\ y_2) \in X \times Y$

$(x_1,y_1) * (x_2,y_2) = (x_1*x_2,\ y_1*y_2)$

Now,

$$\begin{aligned}\eta((x_1,y_1)*(x_2,y_2)) &= \eta(x_1*x_2,y_1*y_2) = f\ (x_1*x_2) + g\ (y_1*y_2)\\ &= (f(x_1)*f(x_2) + (g(y_1) * g(y_2)\\ &= (f(x_1)-f(x_2) + (g(y_1) - g(y_2)\\ &= (f(x_1)+g(y_1)) - (f(x_2) + g(y_2))\\ &= \eta(x_1,y_1) * \eta(x_2,y_2)\end{aligned}$$

$\Longrightarrow \eta((x_1,y_1) * (x_2,y_2)) = \eta(x_1,y_1) * \eta(x_2,y_2)$

Further,

$\eta o u_x(x) = \eta(u_x(x)) = \eta(x,0) = f(x)+g(0) = f(x)+0 = f(x) \ \forall \ x$

$\Longrightarrow \eta o u_x = f$

Similarly $\eta o u_y = g$

Theorem 3.15: Category $\wp_s$ has sums for every pair of objects.

According to P. Freyd [11] and above mentioned results we have the following;

Theorem 3.16: Category $\wp_s$ is an abelian category.

For any $X,Y \in \wp_s$, Hom(X,Y) forms an abelian group where X and Y are considered as abelian groups.

Chapter IV

HOMOLOGICAL TECHNIQUES IN $\mathcal{B}_{CK}$ AND $\mathcal{B}_{CI}$ CATEGORIES

4.1 Introduction: This chapter covers the study of BCK and BCI-homomorphisms through well defined homological techniques. Section 4.2 is devoted to the characterization of different $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ categories in terms of homomorphisms and exact sequences. Existence of first and second isomorphism theorems in these categories is proved in Theorems 4.2 and 4.3 Theorems 4.4 to 4.9 describe the characteristics of morphisms in different $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ categories. In Section 4.3 some homological properties of filters and prime ideals are pointed out. In section 4.4, concept of commutators in $\mathcal{B}_{CK}$ category is introduced and some nice homological results are derived.

4.2 Homological properties and exact sequences

Theorem 4.1: If $f:X \longrightarrow Y$ is a regular morphism in $\mathcal{B}_{CK}$ then $Im(f) \cong Co\text{-}im(f)$.

Proof: Let $f:X \longrightarrow Y$ be a regular morphism in $\mathcal{B}_{CK}$. By the definition of regular morphism Im(f) will be an ideal. Since $\mathcal{B}_{CK}$ is an uniquely factorizable category. So every regular homomorphism $f:X \longrightarrow Y$ in it can be decomposed as $X \xrightarrow{\alpha} Im(f) \xrightarrow{\beta} Y$ with β mono and α an epimorphism.

Let $K \xrightarrow{u} X = ker(f)$ then

Coker(u) = X/K. Thus we can define a projection $p: X \longrightarrow X/K$ such that ker(p) = ker(f).

Now, Coker(u) = X/K

$\Longrightarrow$ Coker(ker(f)) = X/K

$\Longrightarrow$ Co-im(f) = X/K

Consider the diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] & X/K \arrow[dl, dashed, "q"] \\ Y & \end{tikzcd}$$

By prop. 1.21, there exists a unique homomorphism $q: X/K \longrightarrow Y$ s.t. $f = q \circ p$

Ker(p) = Ker(f) $\Longrightarrow$ q is a monomorphism.

Hence we get that $f: X \longrightarrow Y$ can be uniquely factored through as $X \xrightarrow{p} \text{Co-im}(f) \xrightarrow{q} Y$.

By the uniquely factorization theorem we have

Im(f) $\cong$ Co-im(f)

Corollary 4.1: In category $\mathcal{B}_{CK(r)}$, Im(f) $\cong$ Co-im(f).

Lemma 4.1: Let X, Y be BCK-algebras and A be an ideal of X. If $f: X \longrightarrow Y$ be a regular morphism in $\mathcal{B}_{CK}$ then the following are equivalent;

(1) There is a unique homomorphism $\bar{f}: X/A \longrightarrow Y$ s.t. $\bar{f} \circ p = f$ where $p: X \longrightarrow X/A$ is a natural projection

(2) $A \subseteq \ker(f)$, further $\bar{f}$ is a monomorphism if and only if $A = \ker(f)$.

Proof: If (1) holds then

$$\begin{aligned}\ker(f) &= \ker(\bar{f}op)\\ &= p^{-1}[\ker(\bar{f})] \supseteq A\end{aligned}$$

$\Longrightarrow \ker(f) \supseteq A$

Further if f is a monomorphism then $p^{-1}[\ker(\bar{f})] = A$

$\Longrightarrow \ker(f) = A$

(2) $\Longrightarrow$ (1). Since $\ker(f) \supseteq A = \ker(p)$

Hence by proposition 1.21 there exists a unique morphism $\bar{f}: X/A \longrightarrow Y$ s.t. $\bar{f}op = f$.

Theorem 4.2: (First isomorphism theorem). Let X and Y be two BCK-algebras and $f:X \longrightarrow Y$ be a regular epimorphism in $\mathcal{B}_{CK}$ then $Y \cong X/\ker(f)$

Proof: Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r, "u"] & X \arrow[r, "p"] \arrow[d, "f"'] & X/\ker(f) \arrow[r] & 0 \\ & & Y \arrow[ur, "h"'] & & \end{tikzcd}$$

such that the upper row is exact. Since f is an epimorphism and $\ker(p) = \ker(f)$. Therefore by prop.1.21, there exists a unique homomorphism $h:Y \longrightarrow X/\ker(f)$ s.t. $hof = p$.

Further, p is an epimorphism ===> hof is epic ===> h is an epimorphism. Also ker(p) = ker(f) ===> h is a monomorphism. Hence h is an *isomorphism.*

Lemma 4.2: For the sequence

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

in category $\mathcal{B}_{CK(r)}$, the following are equivalent,

(a) The sequence is exact,

(b) $\alpha = \ker(\beta)$ and β is epimorphism,

(c) $\beta = \operatorname{coker}(\alpha)$ and α is monomorphism.

Theorem 4.3: (Second isomorphism theorem).

Let $X \in \mathcal{B}_{CK(r)}$, H and K be the ideals of X such that $H \subseteq K$ then the sequence

$$0 \longrightarrow K/H \xrightarrow{\nu} X/H \xrightarrow{\alpha} X/K \longrightarrow 0$$

is exact.

Proof. Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & 0 \arrow[r] & H \arrow[r, "i"] \arrow[d, "i"'] & K \arrow[r, "p"] \arrow[d, "i"] & K/H \arrow[r] & 0 \\
 & & X \arrow[d, "p"'] & X \arrow[d, "p"] & & \\
0 \arrow[r] & K/H \arrow[r, "\nu"] & X/H \arrow[r, "\alpha"] & X/K \arrow[r] & 0 &
\end{tikzcd}$$

such that all the sequences are exact except the lower one.

We claim that the lower sequence is also exact. Define $\alpha : X/H \longrightarrow X/K$ such that $\alpha(x/H) = x/K \;\forall\; x/H \in X/H$ and $x/K \in X/K$. Clearly α is an epimorphism and $\nu = \ker(\alpha)$. Hence by lemma 4.2 we have that the sequence

$$0 \longrightarrow K/H \xrightarrow{\nu} X/H \xrightarrow{\alpha} X/K \longrightarrow 0$$

is exact.

Lemma 4.3: In an arbitrary exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$$

in $\mathcal{B}_{CK(r)}$ the following are equivalent

(a) f is an epimorphism,

(b) g is a trivial homomorphism,

(c) h is a monomorphism.

Proof: Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ be an arbitrary exact sequence in category $\mathcal{B}_{CK(r)}$ with f as an epimorphism. Then $Im(f) = Y = \ker(g) \implies g(y) = 0 \;\forall\; y \in Y \implies g$ is a trivial morphism in $\mathcal{B}_{CK(r)}$. Hence (a) ===> (b).

Now to show that (b) ===> (c).

Consider g as a trivial homomorphism. Then $g(y) = 0 \;\forall\; y \in Y$. So, $Im(g) = 0 = \ker(h) \implies h$ is a monomorphism.

Lastly to show that (c) ===> (a);

h is a monomorphism ===> ker(h) = 0. Since the sequence is exact, therefore, Im(g) = ker(h) ===> Im(g) = 0. Hence g is a trivial homomorphism ===> ker(g) = Y. Again, as the sequence is exact ===> ker(g) = Im(f) = Y ===> f is an epimorphism.

Theorem 4.4: For an arbitrary exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$$

in the category $\mathcal{B}_{CK(r)}$, C=0 if and only if f is an epimorphism and k is a monomorphism.

Proof: Let C = 0, then g(x) = 0 $\forall$ x $\in$ B ===> ker(g) = B

Due to the exactness of the sequence Im(f) = ker(g) = B

===> f is an epimorphism.

Again C = 0 ===> Im(h) = 0 = ker(k) ===> k is a monomorphism.

Conversely, assume that f is an epimorphism. So

Im(f) = B ===> ker(g) = B ===> g(x) = 0 $\forall$ x $\in$ B.

Hence g is a trivial morphism ===> C = 0.

Now, let k be a monomorphism, then ker(k) = 0 = Im(h) ===> h is a trivial morphism. Thus we have C = 0.

Theorem 4.5: In an arbitrary exact sequence

$$A \xrightarrow{\theta} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{h} E \xrightarrow{k} F$$

in the category $\mathcal{B}_{CK(r)}$, the following are equivalent;

(1) g is an isomorphism,

(2) f and h are trivial,

(3) θ is an epimorphism and k is a monomorphism.

Proof: (1) ===> (2). Let g be an isomorphism, then g is a monomorphism. So $\ker(g) = 0$ ===> $\mathrm{Im}(f) = 0$ ===> f is a trivial morphism.

Again, g is an isomorphism ===> g is an epimorphism. So $\mathrm{Im}(g) = D = \ker(h)$ ===> $h(x) = 0 \ \forall \ x \in D$. Thus we have that h is a trivial morphism.

Now to show that (2) ===> (3). Assume that f is a trivial morphism then $\ker(f) = B$. But the exactness of the sequence gives that $\ker(f) = \mathrm{Im}(\theta)$ ===> $\mathrm{Im}(\theta) = B$ ===> θ is an epimorphism;

Also, h is trivial ===> $\mathrm{Im}(h) = 0$ ===> $\ker(k) = 0$ ===> k is a monomorphism;

Finally to check that (3) ===> (1) i.e. consider θ as an epimorphism and k monomorphism. θ epimorphism ===> $\mathrm{Im}(\theta) = B = \ker(f)$ ===> f is a trivial morphism. Hence $\mathrm{Im}(f) = 0 = \ker(g)$ ===> g is a *monomorphism*.

Again, k is a monomorphism ===> $\ker(k) = 0$. But the sequence is exact ===> $\ker(k) = \mathrm{Im}(h) = 0$ ===> h is a trivial morphism. Thus $\ker(h) = D = \mathrm{Im}(g)$ ===> g is an *epimorphism*. Hence we have g is an isomorphism.

Theorem 4.6: The sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{B}_{CI(r)}$ is exact if and only if

(a) $gof = 0$

(b) if $h: B \longrightarrow Y$ be a morphism in $\mathcal{B}_{CI(r)}$. With $hof = 0$ then there exists a unique homomorphism $\nu : C \longrightarrow Y$ s.t. $\nu og = h$ i.e. the diagram.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] \arrow[dl, dashed, "\nu"] & 0 \\
& Y & &
\end{tikzcd}$$

Commutes.

Proof: Assume that the sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact in $\mathcal{B}_{CI(r)}$. Then $Im(f) = ker(g) \Longrightarrow gof = 0$.

If $h: B \longrightarrow Y$ be a morphism in $\mathcal{B}_{CI(r)}$ with $hof = 0$, then $Im(f) \subseteq ker(h) \Longrightarrow ker(g) \subseteq ker(h)$. The sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact $\Longrightarrow$ g is epic. Hence by prop.1.21 there exists a unique morphism $\nu : C \longrightarrow Y$ s.t. $\nu og = h$.

Conversely if (a) and (b) hold then to show that the sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. By (a) $gof = 0 \Longrightarrow Im(f) \subseteq ker(g)$. Since f is a morphism in the category $\mathcal{B}_{CI(r)}$, so $Im(f)$ is an ideal of $B \Longrightarrow B/Im(f)$

is a BCI-algebra. We, therefore, have a natural morphism $\eta: B \longrightarrow B/Im(f)$ with $\eta o f = 0$.

By (b), there exists a morphism $\nu: C \longrightarrow B/Im(f)$ in $\mathcal{B}_{CI(r)}$ s.t. the diagram.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\eta"'] & C \arrow[r] \arrow[dl, dashed, "\nu"] & 0 \\ & B/Im(f) & & \end{tikzcd}$$

Commutes, i.e. $\nu o g = \eta$.

$ker(g) \subseteq ker\ (\nu o g) = ker\ (\eta) = Im(f) \Longrightarrow ker(g) \subseteq Im(f)$.

Thus we have $Im(f) = ker(g)$.

Now to show that g is an epimorphism.

Let $\eta_1, \eta_2: C \longrightarrow Y$ be morphisms in $\mathcal{B}_{CI(r)}$ such that

$$\eta_1 o g = \eta_2 o g : B \longrightarrow Y$$

Now, we put $h = \eta_1 o g = \eta_2 o g : B \longrightarrow Y$

$\Longrightarrow hof = (\eta_1 o g) o f = \eta_1 o\ (gof) = 0$ [Since gof = 0 by (a)]

By (b) we get that h must be factored through g uniquely. But we have

$$h = \eta_1 o g = \eta_2 o g$$

$\Longrightarrow \eta_1 = \eta_2$. Hence g is epic.

Theorem 4.7. The sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact in $\mathcal{B}_{CI}$ if and only if

(i) $gof = 0$

(ii) If $h: X \longrightarrow B$ is a BCI-homomorphism with $goh = 0$, then there exists a *unique* BCI-homomorphism $n: X \longrightarrow A$ s.t. the diagram

$$\begin{tikzcd} & & X \arrow[dl, dashed, "\eta"'] \arrow[d, "h"] \arrow[dr, dashed, "0"] & \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

Commutes.

Proof: Consider the sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ as an exact sequence, then $Im(f) = ker(g)$ gives that $gof = 0$.

Now the morphism $h: X \longrightarrow B$ with $goh = 0 \Longrightarrow Im(h) \subseteq ker(g) = Im(f) \Longrightarrow Im(h) \subseteq Im(f)$. Since f is monic, so by proposition 1.22, there exists a unique morphism $\eta: X \longrightarrow$. A s.t. the diagram.

$$\begin{tikzcd} & & X \arrow[dl, dashed, "\eta"'] \arrow[d, "h"] \arrow[dr, dashed, "0"] & \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

Commutes.

Conversely if (i) and (ii) hold then to show that the sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact.

(i) gives that $gof = 0 \Longrightarrow Im(f) \subseteq ker(g)$. Suppose $ker(g) = X$. Then the sequence $X \xrightarrow{i} B \xrightarrow{g} C$ is exact, where i is the inclusion map. Thus goi = 0. Hence (ii) implies that there exists a unique morphism $\eta: X \longrightarrow A$ s.t. $fo\eta = i$. So we have,

$$\mathrm{Ker}(g) = \mathrm{Im}(i) = \mathrm{Im}(fo\eta) \subseteq \mathrm{Im}(f)$$

Hence Im(f) = ker(g) ===> the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \text{ is exact at B.}$$

Lastly to show that f is monic. Let there be two morphisms $g_1, g_2: X \longrightarrow A$ such that $fog_1 = fog_2$. On putting $h = fog_1$ we have $goh = (gof)og_1 = 0$ by (ii) we have $g_1 = g_2$ ===> f is a monomorphism.

Theorem 4.8: Let

$$\begin{tikzcd} & X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] \arrow[d, "h"] & Z \arrow[d, "p"] \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

be a commutative diagram in the category $\mathcal{B}_{CK(r)}$ s.t. the upper row is semi-exact and the lower is exact, then there exists a unique morphism $\eta: X \longrightarrow A$ in $\mathcal{B}_{CK(r)}$ s.t. the diagram.

$$\begin{tikzcd} & X \arrow[r, "\alpha"] \arrow[d, dashed, "\eta"'] & Y \arrow[r, "\beta"] \arrow[d, "h"] & Z \arrow[d, "p"] \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

is commutative

Proof: Let the diagram

$$\begin{tikzcd} & X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] \arrow[d, "h"] & Z \arrow[d, "p"] \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

be commutative, then

$$\begin{aligned} (goh)o\alpha &= (po\beta)o\,\alpha \\ &= po\,(\beta o\alpha) \\ &= po0 \quad \text{[Since the upper row is semi-exact]} \\ &= 0 \end{aligned}$$

$\Longrightarrow (goh)o\,\alpha = 0$

$\Longrightarrow go(ho\alpha) = 0$

$\Longrightarrow$ Im$(ho\alpha)$ is contained in ker(g) = Im(f)

Since the sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact $\Longrightarrow$ f is a monomorphism. By prop. 1.22, there exists a unique homomorphism η: X $\longrightarrow$ A such that the diagram

$$\begin{tikzcd} & X \arrow[r, "\alpha"] \arrow[d, dashed, "\eta"'] & Y \arrow[r, "\beta"] \arrow[d, "h"] & Z \arrow[d, "p"] \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

is commutative.

Theorem 4.9. Let the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[r, "g"] \arrow[d, "h"'] & C \arrow[r] & 0 \\ X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] & Z & \end{tikzcd}$$

be commutative in the category $\mathcal{B}_{CK(r)}$ in which upper row is exact and lower row is semi-exact. Then there exists a unique $\mathcal{B}_{CK}$-homomorphism $\theta: C \longrightarrow Z$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] \arrow[d, dashed, "\theta"] & 0 \\ X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] & Z & \end{tikzcd}$$

is commutative.

Proof: Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] & 0 \\ X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] & Z & \end{tikzcd}$$

in the category $\mathcal{B}_{CK(r)}$. Then

$$\beta o(hof) = \beta o(\alpha oq)$$
$$= (\beta o\alpha)oq$$
$$= 0oq \qquad \text{[The sequence } X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \text{ is semi-exact]}$$
$$= 0$$

$\Longrightarrow \beta o(hof) = 0$

$\Longrightarrow (\beta oh)of = 0$

$\Longrightarrow Im(f) \subseteq ker(\beta oh)$.

Since $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact $\Longrightarrow ker(g) \subseteq ker(\beta oh)$ and g is an epimorphism. Thus by prop.1.21 there exists a unique morphism $\theta: C \longrightarrow Z$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] \arrow[d, dashed, "\theta"] & 0 \\ X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] & Z & \end{tikzcd}$$

is commutative

4.3 Some results about filters and prime ideals in $\mathcal{B}_{CK}$ category:

Proposition 4.1: The image of a filter under an epimorphism in the category $\mathcal{B}_{CK(\wedge)}$ is a filter.

Proof: Let X, Y $\in \mathcal{B}_{CK(\wedge)}$ and h: X $\longrightarrow$ Y be an epimorphism. If F is a filter of X then to show that h(F) is a filter of Y.

Assume that $y \in h(F)$ and $y_1 \geq y$. Since h is an epimorphism, there exists $x \in F$ s.t. $y = h(x)$.

Let $x_1 \in h^{-1}(y_1) \neq \emptyset$ in X, then $h(x_1) = y_1$. We claim that it is possible only when $x_1 \geq x$. If not, then $x_1 < x$ $\Longrightarrow h(x_1) < h(x) \Longrightarrow y_1 < y$ which is a contradiction of our supposition $y_1 \geq y$.

Thus, whenever $y \in h(F)$ and $y_1 \geq y \Longrightarrow y_1 \in h(F)$. Now $y_1, y_2 \in h(F) \Longrightarrow$ there exist $x_1, x_2 \in F$ s.t. $h(x_1) = y_1$ and $h(x_2) = y_2$ and glb $(y_1, y_2) = y_1 \wedge y_2 = h(x_1) \wedge h(x_2) = h(x_1 \wedge x_2)$. Since F is a filter, so $x_1 \wedge x_2 \in F \Longrightarrow h(x_1 \wedge x_2) \in h(F) \Longrightarrow$ h(F) is a filter.

Proposition 4.2: The image of a prime ideal in $\mathcal{B}_{CK(\wedge)}$ under an epimorphism is a prime ideal.

Proof: Let X, Y $\in \mathcal{B}_{CK(\wedge)}$ and $f:X \longrightarrow Y$ be an epimorphism. P being a prime ideal of X

$\Longrightarrow$ (X-P) is a filter of X

$\Longrightarrow$ f(X-P) is a filter of Y [By prop.4.1]

$\Longrightarrow$ X-f(P) is a filter of Y,

$\Longrightarrow$ f(P) is a prime ideal of Y.

Hence homomorphic image of a prime ideal under an epimorphism is also a prime ideal.

Proposition 4.3: Pre-image of a filter is also a filter

Proof: Let X, Y $\in \mathcal{B}_{CK(\wedge)}$ and $f:X \longrightarrow Y$ be a homomorphism. If F' is a filter of Y, then to show that $h^{-1}(F')=\{x \mid h(x)\in F'\}$ is a filter of X.

Let $x \in h^{-1}(F')$ and $x_1 \geq x$.

$x_1 \geq x \Longrightarrow h(x_1) \geq h(x) \Longrightarrow h(x_1) \in F'$ (as F' is a filter) $\Longrightarrow x_1 \in h^{-1}(F')$.

$x_1, x_2 \in h^{-1}(F') \Longrightarrow h(x_1), h(x_2) \in F' \Longrightarrow$ $h(x_1) \wedge h(x_2) = h(x_1, \wedge x_2) \in F'$

$\Longrightarrow x_1 \wedge x_2 \in h^{-1}(F') \Longrightarrow h^{-1}(F')$ is a filter.

Proposition 4.4: Pre-image of a prime ideal is a prime ideal.

Proof: Let X, Y $\in \mathcal{B}_{CK(\wedge)}$ and $f:X \longrightarrow Y$ be a homomorphism. If P' is a prime ideal of Y, then $(Y-P')$ is a filter of Y.

$\Longrightarrow f^{-1}(Y-P') = (x-f^{-1}(P'))$ is filter of X

$\Longrightarrow f^{-1}(P')$ is a prime ideal of X.

Hence pre-image of a prime ideal is a prime ideal.

Proposition 4.5: Let $(F_i)_{i \in I}$ be a family of filters where F_i is a filter of M_i, then $\prod F_i$ is a filter of $\prod M_i$.

Proof: Let $f, g \in \prod M_i$ s.t. $f \in \prod F_i$ and $g \geq f$.

$g \geq f \implies g(i) \geq f(i)$ for all $i \in I$. Since F_i is a filter $\implies g(i) \in F_i$ for all $i \in I \implies g \in \prod F_i$. Hence we obtain that $f \in \prod F_i$ and $g \geq f \implies g \in \prod F_i$.

Now, $f, g \in \prod F_i \implies f(i), g(i) \in F_i \ \forall\, i \in I$.

$\implies f(i) \wedge g(i) = (f \wedge g)(i) \in F_i$ (as F_i is a filter)

$\implies (f \wedge g) \in \prod F_i \implies \prod F_i$ is a filter.

Proposition 4.6: If F is a filter of $\prod M_i$, then $F_i = p_i(F)$ is a filter of M_i for all $i \in I$.

Proof: Let $x, y \in M_i$ s.t. $x \in F_i$ and $y \geq x$.

Suppose that $f \in F$ s.t. $p_i(f) = x$.

Let $g : I \longrightarrow \cup M_i$ be a function s.t.

$$g(j) = \begin{cases} f(j) & \text{when } j \neq i \\ y & \text{when } j = i \end{cases}$$

Since Y is a filter of $\prod M_i \implies g \in F$ whenever $g \geq f$.

Hence $p_i(g) = g(i) = y \in F_i \implies$ axiom (F.1) follows.

Now $x, y \in F_i$ implies that there exist $f, g \in F$ s.t. $p_i(f) = x$ and $p_i(g) = y$. Since F is a filter $\implies f \wedge g \in F$ $\implies p_i\,(f \wedge g) \in F_i$.

$\implies p_i(f) \wedge p_i(g) = x \wedge y = \text{glb}\,(x, y) \in F_i$

$\implies F_i$ is a filter of M_i for all $i \in I$.

4.4 Commutators in $\mathcal{B}_{CK}$ category

In this section the concept of commutator ideal in the category $\mathcal{B}_{CK}$ has been introduced and some nice results are obtained.

Let X be an object in the category $\mathcal{B}_{CK}$ and $x_1, x_2 \in X$, then the *commutator* related to the pair (x_1,x_2) denoted by $[x_1,x_2]$ is the set $\{(x_1\wedge x_2)*(x_2\wedge x_1), (x_2\wedge x_1)*(x_1\wedge x_2)\}$ i.e. $[x_1,x_2] = \{(x_1\wedge x_2)*(x_2\wedge x_1),(x_2\wedge x_1)*(x_1\wedge x_2)\}$.

An ideal generated by all the elements of the type $(x_1\wedge x_2) * (x_2\wedge x_1)$ and $(x_2\wedge x_1) * (x_1\wedge x_2)$ is called *commutator ideal* or simply *commutator* of X. We shall denote it by [X,X].

The following results are necessary to study the homological aspects of commutator in $\mathcal{B}_{CK}$ We are omitting the proofs of these results.

Proposition 4.7: An object $X \in \mathcal{B}_{CK}$ is commutative if and only if $[X,X] = \{0\}$.

Proposition 4.8: For $X \in \mathcal{B}_{CK}$, $X/[X,X]$ is a commutative BCK-algebra.

Proposition 4.9: Let $X \in \mathcal{B}_{CK(\wedge)}$ and K be an ideal of X, then $X/K \in \mathcal{B}_{CK(\wedge)}$ if and only if $[X,X] \subseteq K$.

Proposition 4.10: Let $f: X \longrightarrow Y$ be aBCK-homomorphism, then there exists a unique BCK-homomorphism
$[f]: [X,X] \longrightarrow [Y,Y]$ such that the diagram

$$\begin{tikzcd} [X,X] \arrow[r, "{[f]}"] \arrow[d, "i_X"'] & [Y,Y] \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative.

Proof: Let $f: X \longrightarrow Y$ be a BCK-homomorphism and $(x_1 \wedge x_2) * (x_2 \wedge x_1) \in [X,X]$, then

$$f((x_1 \wedge x_2)*(x_2 \wedge x_1) = f(x_1 \wedge x_2) * f(x_2 \wedge x_1)$$
$$= (f(x_1) \wedge f(x_2)) * (f(x_2) \wedge f(x_1))$$

But $(f(x_1) \wedge f(x_2)) * (f(x_2) \wedge f(x_1)) \in [Y,Y]$

$\Longrightarrow$ $f((x_1 \wedge x_2) * (x_2 \wedge x_1)) \in [Y,Y]$

Similarly, it can be shown that

$f((x_2 \wedge x_1)*(x_1 \wedge x_2)) \in [Y,Y]$. Therefore, we have

$f[x_1,x_2] \in [Y,Y]$ for all $x_1,x_2 \in X$

$\Longrightarrow$ $f[X,X] \subseteq [Y,Y]$

Now, by taking the restriction of f to the commutator [X,X] we get,

$$[f]: f/[X,X] : [X,X] \longrightarrow [Y,Y] \text{ s.t.}$$

the diagram

$$\begin{tikzcd} {[X,X]} \arrow[r, dashed, "{[f]}"] \arrow[d, "i_X"'] & {[Y,Y]} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative.

Corollary 4.2: For an identity morphism I_X: X ⟶ X, there exists an identity morphism $[I_X] = I_{[X,X]}$: [X,X] ⟶ [X,X] such that the diagram

$$\begin{tikzcd} {[X,X]} \arrow[r, "{[I_X]}"] \arrow[d, "i_X"'] & {[X,X]} \arrow[d, "i_X"] \\ X \arrow[r, "I_X"] & X \end{tikzcd}$$

is commutative.

Corollary 4.3: If f_1: X ⟶ Y and f_2: Y ⟶ Z be BCK-homomorphisms then

$$[f_2 o f_1] = [f_2] o [f_1]$$

Proof: By prop. 4.10 we have the commutative diagrams.

$$\begin{tikzcd} {[X,X]} \arrow[r, "{[f_1]}"] \arrow[d, "i_X"'] & {[Y,Y]} \arrow[d, "i_Y"] \\ X \arrow[r, "f_1"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {[Y,Y]} \arrow[r, "{[f_2]}"] \arrow[d, "i_Y"'] & {[Z,Z]} \arrow[d, "i_Z"] \\ Y \arrow[r, "f_2"] & Z \end{tikzcd}$$

Now, by pasting the above diagrams together, we get the commutative diagram.

$$\begin{tikzcd}
{[X,X]} \arrow[r, "{[f_1]}"] \arrow[d, "i_X"'] & {[Y,Y]} \arrow[r, "{[f_2]}"] \arrow[d, "i_Y"'] & {[Z,Z]} \arrow[d, "i_Z"] \\
X \arrow[r, "f_1"] & Y \arrow[r, "f_2"] & Z
\end{tikzcd} \quad \ldots(\mathrm{I})$$

and the diagram

$$\begin{tikzcd}
{[X,X]} \arrow[r, "{[f_2 o f_1]}"] \arrow[d, "i_X"'] & {[Z,Z]} \arrow[d, "i_Z"] \\
X \arrow[r, "f_2 o f_1"] & Z
\end{tikzcd} \quad \ldots(\mathrm{II})$$

is also commutative. From the diagrams (I) and (II) we have

$$[f_2 o f_1] = [f_2] o [f_1]$$

Proposition 4.11 : For any BCK-monomorphism $f: X \longrightarrow Y$, the morphism $[f]: [X,X] \longrightarrow [Y,Y]$ is a monomorphism.

Proof: Consider a pair of morphisms $\theta_1, \theta_2: Z \longrightarrow [X,X]$ s.t. the diagram

$$\begin{tikzcd}
Z \arrow[r, shift left, "\theta_1"] \arrow[r, shift right, "\theta_2"'] & {[X,X]} \arrow[r, "{[f]}"] \arrow[d, "i_X"'] & {[Y,Y]} \arrow[d, "i_Y"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

is commutative i.e. $f o i_X o \theta_1 = f o i_X o \theta_2$

$\Longrightarrow$ $i_X o \theta_1 = i_X o \theta_2$ [as f is a monomorphism]

$\Longrightarrow \theta_1 = \theta_2 \Longrightarrow [f]$ is a monomorphism.

Lemma 4.3: Pre-image of a commutator contains a commutator.

Proof: Let $f: X \longrightarrow Y$ be an onto BCK-homomorphism and $y_1, y_2 \in [Y,Y]$, then, $[y_1,y_2] = \{(y_1 \wedge y_2)*(y_2 \wedge y_1), (y_2 \wedge y_1)*(y_1 \wedge y_2)\}$.

Since f is onto $\Longrightarrow \exists\ x_1, x_2, \ldots, x_n \in X$ s.t. $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

$$\begin{aligned} \Longrightarrow (y_1 \wedge y_2) * (y_2 \wedge y_1) &= (f(x_1 \wedge f(x_2) * (f(x_2) \wedge f(x_1)), \\ &= (f(x_1 \wedge x_2) * (f(x_2 \wedge x_1)), \\ &= f((x_1 \wedge x_2) * (x_2 \wedge x_1)). \end{aligned}$$

Similarly, it can be shown that

$$(y_2 \wedge y_1) * (y_1 \wedge y_2) = f((x_2 \wedge x_1) * (x_1 \wedge x_2)).$$

Hence, we have,

$$f^{-1}[y_1, y_2] \supseteq [x_1, x_2]$$

Theorem 4.10: For any BCK-isomorphism $f: X \longrightarrow Y$, the morphism $[f]: [X,X] \longrightarrow [Y,Y]$ is an isomorphism.

Proof: Let $f: X \longrightarrow Y$ be a BCK-isomorphism, then f is an onto BCK-homomorphism.

Consider $a_1, a_2, \ldots, a_n$ as the generators of X, then for any $x \in X$, we have

$$(\ldots(\ldots(x*a_1)*a_2 * \ldots * a_n) = 0$$

Now for any $y \in [Y,Y]$ there exist $b_1, b_2, \ldots b_n$ as the generators of [Y,Y] such that

$$(\ldots(\ldots(y*b_1)*b_2* \ldots * b_n) = 0 \qquad \ldots [5.1]$$

Since f is onto ===> $\exists$ $x \in X$ and $a_1, a_2, \ldots a_n$ as the generators of [X,X] such that $y = f(x)$ and $f(a_1) = b_1, f(a_2) = b_2, \ldots, f(a_n) = b_n$. Hence [5.1] implies that

$$(\ldots(\ldots(f(x)*f(a_1)) * f(a_2) * \ldots * f(a_n) = 0$$

$$\Longrightarrow f(\ldots(\ldots(x*a_1) * a_2 * \ldots * a_n) = 0$$

f is mono ===> $(\ldots(\ldots(x*a_1)*a_2)*\ldots*a_n) = 0$

===> $x \in [X,X]$

===> f is an onto morphism.

By proposition 4.11, [f] is a monomorphism.

Hence [f] is an isomorphism.

Theorem 4.11: For any BCK-homomorphism $f: X \longrightarrow Z$, where Z is a commutative BCK-algebra, the following hold;

(i) $[X,X] \subseteq \ker(f)$

(ii) There exists a unique homomorphism $f': X/[X,X] \longrightarrow Z$ s.t. $f' o p_X = f$ where $p_X: X \longrightarrow X/[X,X]$ is a natural projection.

(iii) $\mathrm{Hom}(X/[X,X], Z) \cong \mathrm{Hom}(X,Z)$.

Proof: Let $f: X \longrightarrow Z$ is a BCK-homomorphism where Z is a commutative BCK-algebra. For any $(x_1 \wedge x_2)*(x_2 \wedge x_1) \in [X,X]$ $f((x_1 \wedge x_2)*(x_2 \wedge x_1)) = f(x_1 \wedge x_2)*f(x_2 \wedge x_1) = (f(x_1) \wedge f(x_2)) * (f(x_2) \wedge f(x_1))$

Since Z is a commutative BCK-algebra and $f(x_1), f(x_2) \in Z$,

===> $f(x_1) \wedge f(x_2) = f(x_2) \wedge f(x_1)$

===> $f(x_1) \wedge f(x_2) \leq f(x_2) \wedge f(x_1)$

$\Longrightarrow (f(x_1) \wedge f(x_2)) * (f(x_2) \wedge f(x_1)) = 0$

$\Longrightarrow (f(x_1 \wedge x_2)) * (f(x_2 \wedge x_1)) = 0$

$\Longrightarrow f((x_1 \wedge x_2) * (x_2 \wedge x_1)) = 0$ [f is a BCK-homomorphism]

Similarly we may obtain, $f((x_2 \wedge x_1) * (x_1 \wedge x_2)) = 0$

Thus we have,

$$(x_1 \wedge x_2) * (x_2 \wedge x_1), (x_2 \wedge x_1) * (x_1 \wedge x_2) \in \ker(f) \ \forall\ x_1, x_2 \in X.$$

Hence $[X,X] \subseteq \ker(f)$.

(ii) Define, $f': X/[X,X] \longrightarrow Z$ s.t. $f'(C_x) = f(x)$.

Firstly, we will show that f' is well defined.

Let $C_{x_1} = C_{x_2}$ where $x_1, x_2 \in X$

$\Longrightarrow x_1 * x_2$ and $x_2 * x_1 \in [X,X]$

$\Longrightarrow x_1 * x_2$ and $x_2 * x_1 \in \ker(f)$ [as $[X,X] \subseteq \ker(f)$]

$\Longrightarrow f(x_1 * x_2) = 0 = f(x_2 * x_1)$

$\Longrightarrow f(x_1) * f(x_2) = 0$ and $f(x_2) * f(x_1) = 0$ [as f is a homo.]

$\Longrightarrow f(x_1) \leq f(x_2)$ and $f(x_2) \leq f(x_1)$,

$\Longrightarrow f(x_1) = f(x_2)$

$\Longrightarrow f'(C_{x_1}) = f'(C_{x_2}) \Longrightarrow f'$ is well defined.

Now to show that f' is a BCK-homomorphism.

Let $C_{x_1}, C_{x_2} \in X/[X,X]$, then

$f'(C_{x_1} * C_{x_2}) = f'(C_{x_1 * x_2}) = f(x_1 * x_2) = f(x_1) * f(x_2)$

$= f'(C_{x_1}) * f'(C_{x_2})$

$\Longrightarrow f'(C_{x_1} * C_{x_2}) = f'(C_{x_1}) * f'(C_{x_2}) \Longrightarrow f'$ is a BCK-homomorphism.

Lastly, to show that f' is unique.

Assume that there exists $f'': X/[X,X] \longrightarrow Z$ s.t. $f''op_X = f \implies f'op_X = f''op_X$. But p_X is an epimorphism.

$\implies f' = f''$

Hence $f': X/[X,X] \longrightarrow Z$ is unique.

(iii) To show that $Hom(X/[X,X], Z) \cong Hom(X,Z)$

Define a correspondence $\eta: Hom(X/[X,X],Z) \longrightarrow Hom(X,Z)$ s.t. $\eta(\theta) = \theta op$ where $p: X \longrightarrow X/[X,X]$ is a natural projection.

We claim that η is a BCK-homomorphism.

Let $\theta_1, \theta_2 \in Hom(X/[X,X],Z)$, then

$\eta(\theta_1 * \theta_2) = (\theta_1 * \theta_2)op$. But $\eta(\theta_1) = \theta_1 op$ and $\eta(\theta_2) = \theta_2 op$

$\implies \eta(\theta_1) * \eta(\theta_2) = (\theta_1 op) * (\theta_2 op)$

Now for any $x \in X$,

$$\begin{aligned}
((\theta_1 * \theta_2)op)(x) &= (\theta_1 * \theta_2)\,(p(x)),\\
&= (\theta_1 * \theta_2)\,(C_x), \quad (\text{as } p(x) \in x/[X,X])\\
&= \theta_1(C_x) * \theta_2(C_x),\\
\implies n(\theta_1 * \theta_2) &= \theta_1(C_x) * \theta_2(C_x), \qquad \ldots [5.2]
\end{aligned}$$

$$\begin{aligned}
\text{Further, } ((\theta_1 op)*(\theta_2 op))(x) &= (\theta_1 op)(x) * (\theta_2 op)(x),\\
&= \theta_1(p(x)) * \theta_2(p(x)),\\
&= \theta_1(C_x) * \theta_2(C_x),\\
\implies \eta(\theta_1) * \eta(\theta_2) &= \theta_1(C_x) * \theta_2(C_x) \qquad \ldots [5.3]
\end{aligned}$$

[5.2] and [5.3] imply that η is a BCK-homomorphism.

θ is one-one,

Let $\eta(\theta_1) = \eta(\theta_2) \Longrightarrow \theta_1 o p = \theta_2 o p$. But p is an epimorphism $\Longrightarrow$ η is *one-one*.

η is onto by (ii)

Hence η is an *isomorphism*.

Corollary 4.4: For any commutative BCK-algebra Z, the BCK-homomorphism f: X $\longrightarrow$ Z is monomorphism implies that X is a commutative BCK-algebra.

Proof: Let f: X $\longrightarrow$ Z is a monomorphism, then

$$\ker(f) = 0 \Longrightarrow [X,X] = 0 \quad (\text{as } [X,X] \subseteq \ker(f))$$

$$\Longrightarrow X \text{ is commutative} \quad [\text{By prop. } 4.7]$$

Corollary 4.5: For any homomorphism f: X $\longrightarrow$ Y in $\mathcal{B}_{CK}$ there exists a homomorphism $\hat{f}$: X/[X,X] $\longrightarrow$ Y/[Y,Y] such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\
X/[X,X] \arrow[r, dashed, "\hat{f}"] & Y/[Y,Y]
\end{tikzcd}$$

is commutative.

Proof: The diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\
X/[X,X] \arrow[r, dashed, "\hat{f}"] & Y/[Y,Y]
\end{tikzcd}$$

can be represented as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[r, "p_Y"] & Y/[Y,Y] \\ X/[X,X] \arrow[urr, dashed, "\hat{f}"'] & & \end{tikzcd}$$

Since Y/[Y,Y] is a commutative BCK-algebra. Therefore by (ii) of Theorem 4.11, there exists a unique homomorphism $\hat{f}$: $X/[X,X] \longrightarrow Y/[Y,Y]$ defined by $\hat{f}(C_x) = C_{f(x)}$ s.t. the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\ X/[X,X] \arrow[r, dashed, "\hat{f}"] & Y/[Y,Y] \end{tikzcd}$$

is commutative.

Corollary 4.6: For any identity BCK-homomorphism I_X: $X \longrightarrow X$, there exists,

I_X: $X/[X,X] \longrightarrow X/[X,X]$ s.t. $I_X\ (C_X) = C_X$ i.e.

$I_X = I_X/[X,X]$.

Theorem 4.12: If f_1: $X \longrightarrow Y$ and f_2: $Y_2 \longrightarrow Z$ are BCK-homomorphisms. Then f_1: $X/[X,X] \longrightarrow Y/[Y,Y]$ and f_2: $Y/[Y,Y] \longrightarrow Z/[Z,Z]$ are BCK-homomorphisms s.t. $f_2 o f_1 = f_2 o f_1$

Proof: The existence of f_1 and f_2 follows from Coro. 4.5.

Now, for any $C_X \in X/[X,X]$

$$f_2 o f_1(C_x) = {}^C f_2 o f_1(x) = {}^C f_2(f_1(x))$$
$$= f_2({}^C f_1(x))$$
$$= f_2(f_1(C_x))$$
$$= f_2 o f_1(C_x)$$

$\Longrightarrow f_2 o f_1 = f_2 o f_1.$

Proposition 4.12: If $f: X \longrightarrow Y$ be an epimorphism in $\mathcal{B}_{CK}$ then $\hat{f}: X/[X,X] \longrightarrow Y/[Y,Y]$ is an epimorphism.

Proof: Let $\theta_1, \theta_2: Y/[Y,Y] \longrightarrow Z$ be two homomorphisms in $\mathcal{B}_{CK}$ such that

$$\theta_1 o \hat{f} = \theta_2 o \hat{f} \qquad \ldots [5.4]$$

Now, consider the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] & \\
X/[X,X] \arrow[r, "\hat{f}"] & Y/[Y,Y] \arrow[r, shift left, "\theta_1"] \arrow[r, shift right, "\theta_2"'] & Z
\end{tikzcd}$$

in which the square is commutative by Cor.4.5

From [5.4] we get

$$(\theta_1 o \hat{f}) o p_X = (\theta_2 o \hat{f}) o p_X$$
$$\Longrightarrow \theta_1 o (\hat{f} o p_X) = \theta_2 o (\hat{f} o p_X)$$
$$\Longrightarrow \theta_1 o (p_Y o f) = \theta_2 o (p_Y o f) \qquad [\text{ By Cor } 4.5]$$
$$\Longrightarrow (\theta_1 o p_Y) o f = (\theta_2 o p_Y) o f$$

But f is an epimorphism $\Longrightarrow \theta_1 o p_Y = \theta_2 o p_Y$

Further p_Y is an epimorphism $\Longrightarrow \theta_1 = \theta_2$

Hence $\hat{f}$ is an epimorphism.

Proposition 4.13: For any onto BCK-homomorphism $f: X \longrightarrow Y$, the homomorphism $\hat{f}: X/[X,X] \longrightarrow Y/[Y,Y]$ is an onto morphism.

Proof: Assume $f: X \longrightarrow Y$ is an onto BCK-homomorphism.

Then for any $y \in Y \exists x \in X$ s.t. $y = f(x)$.

Now for any $C_x \in Y/[Y,Y]$, we have

$C_y = C_{f(x)} = \hat{f}(C_x) \Longrightarrow C_y = \hat{f}(C_x) \ \forall\ C_y \in Y/[Y,Y]$

$\Longrightarrow \hat{f}$ is onto.

Proposition 4.14: If $f: X \longrightarrow Y$ is an isomorphism in $\mathcal{B}_{CK}$ then $\hat{f}: X/[X,X] \longrightarrow Y/[Y,Y]$ is an isomorphism.

Proof: Assume that $f: X \longrightarrow Y$ is an isomorphism.

Then $\hat{f}$ is mono and onto both.

$\hat{f}$ is a onto morphism follows from proposition 4.13. Now, we have to show that $\hat{f}$ is mono. For this consider the diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\ X/[X,X] \arrow[r, "\hat{f}"] & Y/[Y,Y] \end{tikzcd}$$

Which is commutative by Cor. 4.5. Therefore,

$\ker(\hat{f}op_X) = \ker\ (p_Y of)$

$$= f^{-1}(\ker(p_Y))$$

$$= f^{-1}[Y,Y]$$

$$= [X,X] \qquad (f \text{ is iso.} \implies [X,X] \cong [Y,Y])$$

$$\implies \ker(\hat{f} o p_X) = [X,X] = \ker(p_X)$$

$$\implies \ker(\hat{f}) = 0 \implies \hat{f} \textit{ is a monomorphism}$$

Hence $\hat{f}$ is an *isomorphism.*

Chapter V

FUNCTORIAL PROPERTIES OF BCK AND BCI-STRUCTURES

5.1 Introduction: This chapter is concerned with the study of functorial properties of some well defined BCK and BCI-structures. A structure has functorial property if there is a functor defined by it.

In this chapter several functors have been constructed through different notions on $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ categories. In section 5.2 we have constructed functors with the help of p-radical of BCI-algebras and discussed their characteristics. In section 5.3 a co-variant functor by retractions and contravariant functor with the help of co-retraction morphisms in some $\mathcal{B}_{CK}$ sub-categories are described. With the help of filters and prime ideals a functor is constructed in section 5.4. In section 5.5 a functor is constructed over $\mathcal{B}_{CK(i_+)}$ category. Using involutions in BCK-algebra a functor is defined in section 5.6. Further a contravariant functor $F: \mathcal{S} \longrightarrow \mathcal{B}_{CK(1,\wedge)}$ is constructed. In section 5.7 Hom functors in $\mathcal{B}_{CK}$ and $\mathcal{B}_{CI}$ categories are described. In section 5.8 functors are constructed by using p-semi simple property of BCI-algebras.

Lastly in section 5.9 we have obtained some interesting functors and natural isomorphisms by using the idea of commutator ideal in $\mathcal{B}_{CK}$ category.

5.2 Functors of p-radicals

Definition 5.1: Let X be a BCI-algebra. The p-radical of X be the set $X_+ = \{x \in X \mid x \geq 0\}$. The p-radical X_+ is an ideal of X and it is a BCK-algebra.

Lemma 5.1: If $f: X \longrightarrow Y$ be a homomorphism in $\mathcal{B}_{CI}$ then $f_+: X_+ \longrightarrow Y_+$ defined by $f_+(x) = f(x)$ is a homomorphism in $\mathcal{B}_{CK}$.

Lemma 5.2: If $I_X: X \longrightarrow X$ is the identity homomorphism then $I_{X_+}: X_+ \longrightarrow X_+$ is also identity homomorphism.

Lemma 5.3: If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are homomorphisms in $\mathcal{B}_{CI}$ then $f_+: X_+ \longrightarrow Y_+$ and $g_+: Y_+ \longrightarrow Z_+$ are the homomorphisms in $\mathcal{B}_{CK}$ and $(gof)_+: X_+ \longrightarrow Z_+$ be s.t. $(gof)_+ = g_+of_+$.

Proof:
$$\begin{aligned}(gof)_+(x) &= (gof)(x)\\ &= g[f(x)]\\ &= g[f_+(x)] \qquad\qquad [\text{since } f_+(x) = f(x)]\\ &= g_+[f_+(x)]\\ &= (g_+of_+)(x) \quad \forall\, x \in X_+\end{aligned}$$

Hence we have, $(gof)_+ = g_+of_+$

Using lemmas 5.1, 5.2 and 5.3 we can define a covariant functor;

$$F_+: \mathcal{B}_{CI} \longrightarrow \mathcal{B}_{CK} \text{ s.t.}$$
$$F_+(X) = X_+ \;\forall\, X \in \mathcal{B}_{CI}$$
$$\text{and } \; F_+(f) = f_+ \;\forall\, f \in \mathcal{B}_{CI}$$

Proposition 5.1: If $f: X \longrightarrow Y$ is a monomorphism then $f_+: X_+ \longrightarrow Y_+$ is a monomorphism.

Proof: Let $f: X \longrightarrow Y$ be a monomorphism and $g_1, g_2: M \longrightarrow X_+$ be a pair of BCK-homomorphism s.t. $f_+ o g_1 = f_+ o g_2$. Consider the inclusion maps $X_+ \xrightarrow{i} X$ and $Y_+ \xrightarrow{i} Y$ represented by the same symbol i, then

$$(f o i) o g_1 = (i o f_+) o g_1 = (i o f_+) o g_2 = (f o i) o g_2$$

$$\Longrightarrow f o (i o g_1) = f o (i o g_2)$$

$$\Longrightarrow i o g_1 = i o g_2 \qquad \text{[As f is mono.]}$$

$$\Longrightarrow g_1 = g_2 \qquad \text{[i is an inclusion map]}$$

So $f_+ o g_1 = f_+ o g_2 \Longrightarrow g_1 = g_2 \Longrightarrow f_+$ is a monomorphism.

Corollary 5.1: The functor $F_+: \mathcal{B}_{CI} \longrightarrow \mathcal{B}_{CK}$ is a mono functor.

Since for any BCI-algebra X, the p-radical X_+ of X is an ideal. So we can always form a quotient BCI-algebra X/X_+.

Proposition 5.2: If $f: X \longrightarrow Y$ is a BCI-homomorphism then $f(X_+) \subseteq Y_+$ and the mapping $\tilde{f}: X/X_+ \longrightarrow Y/Y_+$ defined by $\tilde{f}(C_x) = C_{f(x)}$ is a BCI-homomorphism.

Proof: Let $C_{x_1}, C_{x_2} \in X/X_+$ be any two elements, then

$$\begin{aligned}
\tilde{f}(C_{x_1} * C_{x_2}) &= \tilde{f}(C_{x_1 * x_2}) \qquad \text{[since } C_{x_1} * C_{x_2} = C_{x_1 * x_2}\text{)} \\
&= C_{f(x_1 * x_2)} \\
&= C_{f(x_1) * f(x_2)} \quad \text{[since f is a homo.]} \\
&= C_{f(x_1)} * C_{f(x_2)} \\
&= \tilde{f}(C_{x_1}) * \tilde{f}(C_{x_2})
\end{aligned}$$

Thus $\tilde{f}(C_{x_1} * C_{x_2}) = \tilde{f}\ (C_{x_1}) * \tilde{f}(C_{x_2})$ implies that $\tilde{f}$ is a BCI-homomorphism.

Corollary 5.2: If I_X: $X \longrightarrow X$ is the identity homomorphism in $\mathcal{B}_{CI}$, then I_X: $X/X_+ \longrightarrow X/X_+$ is also identity homomorphism.

Lemma 5.4: If f: $X \longrightarrow Y$ and g: $Y \longrightarrow Z$ are BCI-homomorphisms, then

$$\widetilde{gof} = \tilde{g}o\tilde{f}$$

Proof: Let $C_x \in X/X_+$ be any element, then

$$\begin{aligned}\widetilde{gof}(C_x) &= C_{gof(x)} \\ &= C_{g(f(x))} \\ &= \tilde{g}(C_{f(x)}) \\ &= \tilde{g}(\tilde{f}(C_x) \\ &= \tilde{g}o\tilde{f}(C_x)\end{aligned}$$

$$\Longrightarrow \widetilde{gof} = \tilde{g}o\tilde{f}$$

Now by using Prop. 5.2, Lemma 5.4 and Corollary 5.2 we can define a covariant functor;

$$\tilde{F}: \mathcal{B}_{CI} \longrightarrow \mathcal{B}_{CI} \quad \text{s.t.}$$

$$\tilde{F}(X) = X/X_+$$

and $$\tilde{F}(f) = \tilde{f}$$

Proposition 5.3: The functor $\tilde{F}$: $\mathcal{B}_{CI} \longrightarrow \mathcal{B}_{CI}$ is an epi-functor.

Proof: Let f: $X \longrightarrow Y$ be an epimorphisms. Then for any $y \in Y$, there exists $x \in X$ s.t. $f(x) = y$.

Now let $\tilde{f}: X/X_+ \longrightarrow Y/Y_+$ and choose any $C_y \in Y/Y_+$ then f being an epimorphisms. We have $C_y = C_{f(x)} = \tilde{f}(C_x) \Longrightarrow$ for any $C_y \in Y/Y_+$ there exists $C_x \in X/X_+$ s.t. $\tilde{f}(C_x) = C_{f(x)} = C_y$ Hence $\tilde{f}: X/X_+ \longrightarrow Y/Y_+$ is an onto morphism $\Longrightarrow \tilde{F}: \mathcal{B}_{CI} \longrightarrow \mathcal{B}_{CI}$ is an epi-functor.

5.3 FUNCTORS BY RETRACTIONS AND CO-RETRACTIONS

If X is an associative BCI-algebra then End(X) is also a BCI-algebra.

Consider the category of all BCI-algebras in which morphisms between then are retractions. We denote this category by $\mathcal{B}_{(r)}$.

We can define a functor

$T: \mathcal{B}_{(r)} \longrightarrow \mathcal{B}_{CI}$ such that

$T(X) = \text{End}(X)$ and

for any morphism $f: X \longrightarrow Y$ in $\mathcal{B}_{(r)}$

$T_f: \text{End}(X) \longrightarrow \text{End}(Y)$ in $\mathcal{B}_{CI}$

defined by, $T_f(\theta) = f o \theta o f'$ where $f: X \longrightarrow Y$ is a retraction and $\theta \in \text{End}(X)$.

Lemma 5.5: If $\theta_1, \theta_2 \in \text{End}(X)$ then $\theta_1 * \theta_2 \in \text{End}(X)$

Proof: Let $\theta_1: X \longrightarrow X$ and $\theta_2: X \longrightarrow X$ and $x_1, x_2 \in X$ then $x_1 * x_2 \in X$, then

$$
\begin{aligned}
(\theta_1*\theta_2)(x_1*x_2) &= \theta_1(x_1*x_2)*\theta_2(x_1*x_2) \\
&= [\theta_1(x_1)*\theta_1(x_2)] * [\theta_2(x_1)*\theta_2(x_2)] \\
&= [\theta_1(x_1)*(\theta_2(x_1)*\theta_2(x_2))] * \theta_1(x_2)
\end{aligned}
$$

$= [(\theta_1(x_1)*\theta_2(x_1))] * \theta_2(x_2)] * \theta_1(x_2)$ (by B2)

$= [(\theta_1(x_1)*\theta_2(x_1))] * \theta_1(x_2)] * \theta_2(x_2)$ by(P5.3

$= [\theta_1(x_1)*\theta_2(x_1)] * (\theta_1(x_2)*\theta_2(x_2))$

$= (\theta_1*\theta_2)(x_1) * (\theta_1*\theta_2)(x_2)$

$\Longrightarrow (\theta_1*\theta_2)(x_1*x_2) = (\theta_1*\theta_2)(x_1) * (\theta_1*\theta_2)(x_2)$

$\Longrightarrow (\theta_1*\theta_2): X \longrightarrow X$ is a BCI-homomorphism i.e. $\theta_1*\theta_2 \in End(X)$.

Proposition 5.4: If $f: X \longrightarrow Y$ is a retraction i.e. there exists f': $Y \longrightarrow X$ s.t. $fof' = I_Y$. Then the mapping T_f: $End(X) \longrightarrow End(Y)$ defined by $T_f(\theta) = fo\theta of'$ is a BCI-homomorphism.

Proof: Consider $\theta_1, \theta_2: X \longrightarrow X$ then

$T_f(\theta_1*\theta_2) = fo(\theta_1*\theta_2)of'$ and

$T_f(\theta_1)*T_f(\theta_2) = (fo\theta_1 of')*(fo\theta_2 of')$

Clearly $T_f(\theta_1*\theta_2): Y \longrightarrow Y$, then for any $y \in Y$

$T_f(\theta_1*\theta_2)\ (y) = (fo(\theta_1*\theta_2)of')(y)$

$= (fo(\theta_1*\theta_2)(f'(y))$

$= f[(\theta_1*\theta_2)(f'(y))]$ as $f'(y) \in X$

$= f[\theta_1(f'(y)) * \theta_2\ (f'(y))]$

$= (fo\theta_1 of')(y)*(fo\theta_2 of')(y)$

$= T_f(\theta_1)(y) * T_f(\theta_2)(y)$

$\Longrightarrow T_f(\theta_1*\theta_2)(y) = T_f(\theta_1)(y) * T_f(\theta_2)(y) \ \forall \ y \in Y$

$\Longrightarrow T_f(\theta_1*\theta_2) = T_f(\theta_1) * T_f(\theta_2)$

$\Longrightarrow T_f$ is a BCI-homomorphism.

Proposition 5.5: If $f_1 : X \longrightarrow Y$ and $f_2 : Y \longrightarrow Z$ are retractions then $f_2 o f_1 : X \longrightarrow Z$ is also a retraction.

Proof: Let $f_1 : X \longrightarrow Y$ and $f_2 : Y \longrightarrow Z$ are retractions then there exist $f_1' : Y \longrightarrow X$ and $f_2' : Z \longrightarrow Y$ s.t. $f_1 o f_1' = I_Y$ and $f_2 o f_2' = I_Z$.

Now,

$$
\begin{aligned}
(f_2 o f_1) o (f_1' o f_2') &= f_2 (f_1 o f_1') o f_2' \\
&= f_2 o I_y o f_2' \\
&= f_2 o f_2' \\
&= I_Z
\end{aligned}
$$

$\Longrightarrow (f_2 o f_1) o (f_1' o f_2') = I_Z$

$\Longrightarrow f_2 o f_1 : X \longrightarrow Z$ is a retraction.

Proposition 5.6: If $f_1 : X \longrightarrow Y$ and $f_2 : Y \longrightarrow Z$ are morphisms in $\mathcal{B}_{(r)}$ then ${}^T f_2 o f_1 = {}^T f_2 o {}^T f_1$

Proof: Let $f_1 : X \longrightarrow Y$ and $f_2 : Y \longrightarrow Z$ be retractions i.e. morphisms in the category $\mathcal{B}_{(r)}$. Then ${}^T f_2 o f_1 : End(X) \longrightarrow End(Z)$ is a BCI-homomorphism.

For any $\theta \in Hom(X)$

$$
\begin{aligned}
{}^T f_2 o f_1 (\theta) &= (f_2 o f_1) o \theta o (f_1' o f_2') \\
&= f_2 (f_1 o \theta) o (f_1' o f_2') \\
&= f_2 (f_1 o \theta o f_1') o f_2' \\
&= f_2 o ({}^T f_1 (\theta)) o f_2' \\
&= {}^T f_2 (T f_1 (\theta)) \\
&= ({}^T f_2 o {}^T f_1) (\theta)
\end{aligned}
$$

$\Longrightarrow {}^{T}f_2 o {}^{T}f_1(\theta) = ({}^{T}f_2 o {}^{T}f_1)(\theta) \quad \forall\ \theta \in End(X)$

$\Longrightarrow {}^{T}f_2 o f_1 = {}^{T}f_2 o {}^{T}f_1$

Corollary 5.3: If I_X: X ——> X is an identity morphism in $\mathcal{B}_{(r)}$ then ${}^{T}I_X$: End(X) ——> End(X) is an identity morphism in $\mathcal{B}_{CI}$.

Thus, by proposition 5.4, 5.5, 5.6 and Corallary 5.3 we have T: $\mathcal{B}_{(r)} \longrightarrow \mathcal{B}_{CI}$ is a functor.

Consider the category $\mathcal{B}_{(cr)}$ in which the objects of the category are associative BCI-algebras and morphisms between them are those BCI-homomorphism which are co-retractions.

Lemma 5.6: If f_1: X ——> Y and f_2: Y ——> Z are co-retractions then $f_2 o f_1$: X ——> Z is a co-retraction.

Proposition 5.7: If f:X ——> Y is a morphism in the category $\mathcal{B}_{(cr)}$ then the mapping G_f: End(Y) ——> End(X) defined by $G_f(\beta) = f' o \beta o f$ is a BCI-homomorphism.

Proof: For any morphism f:X ——> Y in $\mathcal{B}_{(cr)}$ the morphism G_f: End(Y) ——> End(X) implies that $G_f(\beta) \in End(X)\ \forall\ \beta \in End(Y)$.

Moreover, $\beta_1, \beta_2 \in End(Y) \Longrightarrow \beta_1 * \beta_2 \in End(Y)$.

For any $x \in X$, we have

$$G_f(\beta_1 * \beta_2)(x) = (f' o(\beta_1 * \beta_2) o f)(x)$$
$$= [f' o(\beta_1 * \beta_2)](f(x))$$

$= f^{\prime}[\beta_1(f(x)) * \beta_2(f(x)]$

$= f^{\prime}(\beta_1(f(x)) * f^{\prime}(\beta_2(f(x))$

$= (f^{\prime}o\beta_1 of)(x) * (f^{\prime}o\beta_2 of)(x)$

$= G_f(\beta_1)(x) * G_f(\beta_2)(x)$

$\Longrightarrow G_f(\beta_1 * \beta_2)(x) = G_f(\beta_1)(x) * G_f(\beta_2)(x) \ \forall \ x \in X$

$\Longrightarrow G_f(\beta_1 * \beta_2) = G_f(\beta_1) * G_f(\beta_2)$

$\Longrightarrow G_f$ is a BCI-homomorphism.

Corollary 5.4: For an identity morphism I_X: $X \longrightarrow X$ in $\mathcal{B}_{(cr)}$, the mapping ${}^{G}I_X$: $End(X) \longrightarrow End(X)$ is an identity morphism.

Proposition 5.8: If f_1: $X \longrightarrow Y$ and f_2: $Y \longrightarrow Z$ are homomorphisms in $\mathcal{B}_{(cr)}$ then ${}^{G}f_2 of_1 = {}^{G}f_1 o {}^{G}f_2$.

Proof: Let f_1: $X \longrightarrow Y$ and f_2: $Y \longrightarrow Z$ be homomorphisms in $\mathcal{B}_{(cr)}$ then ${}^{G}f_2 of_1$: $End(Z) \longrightarrow End(X)$ is a BCI-homomorphism.

For any $\nu \in End(Z)$, we have

${}^{G}f_2 of_1(\nu) = (f_1^{\prime} of_2^{\prime}) o \nu o (f_2 of_1)$ [f_1 and f_2 are co-retractions]

$= f_1^{\prime} of_2^{\prime} o(\nu of_2) of_1$

$= f_1^{\prime} o(f_2^{\prime} o \nu of_2) of_1$

$= f_1^{\prime}({}^{G}f_2(\nu)) of_1$

$= {}^{G}f_1({}^{G}f_2(\nu))$

$= ({}^{G}f_1 o {}^{G}f_2)(\nu)$

$\Longrightarrow {}^{G}f_2 of_1(\nu) = ({}^{G}f_1 o {}^{G}f_2)(\nu) \ \forall \nu \in End(Z)$

$\Longrightarrow G_{f_2 o f_1} = G_{f_1} o G_{f_2}$

With the help of the propositions 5.7, 5.8 and corollary 5.4 we can define a *contravariant functor*.

$G: \mathcal{B}_{(cr)} \longrightarrow \mathcal{B}_{CI}$ s.t.

$G(X) = End(X)$

and for any morphism $f:X \longrightarrow Y$ in $\mathcal{B}_{(cr)}$

$G_f: End(Y) \longrightarrow End(X)$ defined as

$G_f(\beta) = f' o \beta o f$.

5.4 FUNCTORS BY FILTERS AND PRIME IDEALS

If P_1, P_2 and P_3 be the prime ideals of a bounded commutative BCK-algebra X then $(X-P_1)$, $(X-P_2)$ and $(X-P_3)$ will be the corresponding filters of X. Thus there is a one to one correspondence between the ideals and filters of a bounded commutative BCK-algebra. Moreover, if P is a prime ideal of X and $x \in P$ then $N_x \in (X-P)$, the corresponding filter of the ideal P.

Let X be a bounded commutative BCK-algebra and P(X) represents the category of all prime ideals of X. The objects of the category P(X) are the prime ideals and morphisms be the BCK-homomorphisms between them. F(X) denotes the category of filters of X in which objects are the filters and morphisms of the category be the functions between them.

Let P_1, P_2 be the prime ideals of the bounded commutative BCK-algebra X. For any BCK-homomorphism $f:P_1 \longrightarrow P_2$ in P(X), we can define a function

S_f: $(X-P_1) \longrightarrow (X-P_2)$ in $F(X)$ s.t.

$S_f(N_x) = N_{f(x)}$.

Proposition 5.9: If $f: P_1 \longrightarrow P_2$ and $g: P_2 \longrightarrow P_3$ are BCK-homomorphisms where P_1, P_2 and P_3 are prime ideals of the bounded commutative BCK-algebra X, then $S_{gof} = S_g o S_f$.

Proof: Let X be a bounded commutative BCK-algebra. $f: P_1 \longrightarrow P_2$ and $g: P_2 \longrightarrow P_3$ be the morphisms in P(X), then for the composite morphism $gof: P_1 \longrightarrow P_3$ in P(X) the mapping S_{gof}: $(X - P_1) \longrightarrow (X-P_3)$ is a morphism in F(x). For any $N_x \in (X-P_1)$,

$$\begin{aligned} S_{gof}(N_x) &= N_{gof(x)} \\ &= N_{g(f(x))} \\ &= S_g(N_{f(x)}) \\ &= S_g(S_f(N_x)) \\ &= (S_g o S_f)(N_x) \end{aligned}$$

$$\Longrightarrow S_{gof}(N_x) = (S_g o S_f)(N_x) \quad \forall\, N_x \in (X - P_1)$$

Hence $S_{gof} = S_g o S_f$.

Corollary 5.5: If $I_P: P \longrightarrow P$ be the identity homomorphism in P(X) then the map S_{I_P}: $(X-P) \longrightarrow (X-P)$ is also identity morphism in F(X).

In the light of the above discussion we can define a covariant functor.

$S: P(X) \longrightarrow F(X)$ s.t.

$S(P) = (X-P)$

and for any morphism $f:P_1 \longrightarrow P_2$ in $P(X)$

$$S(f) = S_f: (X-P_1) \longrightarrow (X-P_2) \text{ in } F(X).$$

5.5 FUNCTORS BY SELF MAPS

Let X be a positive implicative BCK-algebra. A self map $\ell_x: X \longrightarrow X$ defined as

$$\ell_x(t) = x*t \ \forall \quad t \in X$$

is called a left map of X. The composition of left maps is defined as

$$\ell_x \circ \ell_y = \ell_{x*y}.$$

If X is a positive implicative BCK-algebra then L(X), the collection of all left maps on X is also a positive implicative BCK-algebra.

Lemma 5.7: If $f:X \longrightarrow Y$ is a BCK-homomorphism then the map $L_f: L(X) \longrightarrow L(Y)$ defined by $L_f(\ell_x) = \ell_{f(x)}$ is a BCK-homomorphism.

Proof: Let $f:X \longrightarrow Y$ be a BCK-homomorphism in the category of positive implicative BCK-algebras and $L_f: L(X) \longrightarrow L(Y)$ be a map defined by $L_f(\ell_x) = \ell_{f(x)}$. For any two elements $\ell_{x_1}, \ell_{x_2} \in L(X)$

$$\begin{aligned} L_f(\ell_{x_1} \circ \ell_{x_2}) &= L_f(\ell_{x_1 * x_2}) \\ &= \ell_{f(x_1 * x_2)} \\ &= \ell_{f(x_1) * f(x_2)} \\ &= \ell_{f(x_1)} \circ \ell_{f(x_2)} \end{aligned}$$

[By the definition of composition of maps in L(X)]

$$= L_f(\ell_{x_1}) \circ L_f(\ell_{x_2})$$

$\Longrightarrow L_f(\ell_{x_1} \circ \ell_{x_2}) = L_f(\ell_{x_1}) \circ L_f(\ell_{x_2}) \ \forall\ \ell_{x_1}, \ell_{x_2} \in L(X)$.

$\Longrightarrow L_f: L(X) \longrightarrow L(Y)$ is a homomorphism.

Proposition 5.10: If $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ are BCK-homomorphisms in $\mathcal{B}_{CK(i_+)}$ then

$$L_{gof} = L_g o L_f$$

Proof: Let $f:X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be the morphisms in the category $\mathcal{B}_{CK(i_+)}$ hen the maps $L_f: L(X) \longrightarrow L(Y)$, $L_g: L(Y) \longrightarrow L(Z)$ and their composition $L_{gof}:L(X) \longrightarrow L(Z)$ will be the homomorphisms in the category $\mathcal{B}_{CK(i_+)}$.

For any $\ell_x \in L(X)$

$$\begin{aligned} L_{gof}(\ell_x) &= \ell_{gof(x)} \\ &= \ell_{g(f(x))} \\ &= L_g(\ell_{f(x)}) \\ &= L_g(L_f(\ell_x)) \\ &= (L_g o L_f)(\ell_x) \end{aligned}$$

$\Longrightarrow L_{gof}(\ell_x) = (L_g o L_f)(\ell_x) \ \forall\ \ell_x \in L(X)$

Hence $L_{gof} = L_g o L_f$.

Corollary 5.6: If $I_X: X \longrightarrow X$ be the identity homomorphism then $L_{I_X}: L(X) \longrightarrow L(X)$ is also identity homomorphism.

Now by using Lemma 5.7, Prop. 5.10 and Cor. 5.6 we can define a functor

$L: \mathcal{B}_{CK(i_+)} \longrightarrow \mathcal{B}_{CK(i_+)}$ s.t.

$X \longrightarrow L(X)$

and for any BCK-homomorphism $f:X \longrightarrow Y$

$L(f) = L_f: L(X) \longrightarrow L(Y)$.

5.6 FUNCTOR BY INVOLUTIONS

Let X be a bounded BCK-algebra. The set $I(X) = \{x \in X \mid NN_x = x\}$ is called the set of all involutions of X. If $f:X \longrightarrow Y$ is a BCK-homomorphism from bounded BCK-algebra X to a bounded BCK-algebra Y then we can define an induced map

$f:I(X) \longrightarrow I(Y)$ s.t. $f(I(X)) \subseteq I(Y)$

i.e. homomorphic image of an involution is an involution.

Hence we can define a functor

$$I: \mathcal{B}_{CK(1)} \longrightarrow \mathcal{B}_{CK(1)} \text{ s.t.}$$
$$X \longrightarrow I(X)$$

and for any BCK-homomorphism $f:X \longrightarrow Y$ $I(f): I(X) \longrightarrow I(Y)$ defined by $I(f(x)) = NN(f(x))$.

As we know that [0,1] is a bounded commutative BCK-algebra under the operation defined as

$$x*y = \begin{cases} 0 \text{ if } x \leq y \\ x-y \text{ if } x>y \end{cases}$$

Consider $[0,1]^X$, the set of all functions from the set X to [0,1]. It is also a bounded commutative algebra under the operation.

$$(f*g)(x) = f(x) * g(x)$$

Where f*g is a morphism in $[0,1]^X$

Thus we can define a functor,

$$F: \mathcal{S} \longrightarrow \mathcal{B}_{CK(1,\wedge)} \text{ s.t.}$$

$$F(X) = [0,1]^X$$

For any morphism $f:X \longrightarrow Y$ in $\mathcal{S}$

$$F(f): [0,1]^Y \longrightarrow [0,1]^X$$

This functor F is a *contravariant functor.*

5.7 HOM FUNCTORS:

E.Y. Debba and S.K. Goel in [9] have shown that if X is a BCI-algebra and Y a BCK-algebra then Hom(X,Y) is a BCK-algebra. With the help of this result we can define following functors.

For any BCI-algebra X, there is covariant functor

$$\mathrm{Hom}(X,-): \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$$

such that

$\mathrm{Hom}(X,-)(Y) = \mathrm{Hom}(X,Y)$ for all $Y \in \mathcal{B}_{CK}$ and any morphism $f:Y_1 \longrightarrow Y_2$ in $\mathcal{B}_{CK}$.

$$\mathrm{Hom}(X,-)(f): \mathrm{Hom}(X,f): \mathrm{Hom}(X,Y_1) \longrightarrow \mathrm{Hom}(X,Y_2)$$

For any BCK-algebra Y, there is a contravariant functor.

$$\mathrm{Hom}(-,Y): \mathcal{B}_{CI} \longrightarrow \mathcal{B}_{CK}$$

such that

$\mathrm{Hom}(-,Y)(X) = \mathrm{Hom}(X,Y)$ for all $X \in \mathcal{B}_{CI}$ and for any morphism $g:X_2 \longrightarrow X_1$ in $\mathcal{B}_{CI}$.

$\mathrm{Hom}(-,Y)\ (g) = \mathrm{Hom}(g,Y)$: $\mathrm{Hum}(X_1,Y) \longrightarrow \mathrm{Hom}(X_2,Y)$

we can also define a bifunctor

$\mathrm{Hom}(-,-): \mathcal{B}_{CI} \times \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$

s.t. $\mathrm{Hom}(-,-)(X,Y) = \mathrm{Hom}(X,Y)$ for all $X \in \mathcal{B}_{CI}$ and $Y \in \mathcal{B}_{CK}$ and morphisms $f:Y_1 \longrightarrow Y_2$ in $\mathcal{B}_{CK}$ and $g:X_2 \longrightarrow X_1$ in $\mathcal{B}_{CI}$.

$\mathrm{Hom}(-,-)\ (g,f) = \mathrm{Hom}(g,f)$: $\mathrm{Hom}(X_1,Y_1) \longrightarrow \mathrm{Hom}(X_2,Y_2)$.

This functor is contravariant in first and covariant in second variable.

5.8 FUNCTORS BY p-SEMI SIMPLE PROPERTY OF BCI-ALGEBRAS

Let X in a BCI-algebra and R(X) be the p-radical of X then X/R(X) is a p-semi simple BCI-algebra. For any BCI-algebra X we can always define a projection $p:X \longrightarrow X/R(X)$.

Proposition 5.1: For any homomorphism $f:X \longrightarrow Y$ and projections $p: X \longrightarrow X/R(X)$ and $Y \longrightarrow Y/R(Y)$, there exists a unique map $\bar{f}: X/R(X) \longrightarrow Y/R(Y)$ s.t. the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "p"] & Y/R(Y) \\
X/R(X) \arrow[urr, dashed, "\bar{f}"'] & &
\end{tikzcd}$$

is commutative

Proof: Let $x \in R(X)$, $\Longrightarrow 0 \leq x \Longrightarrow 0 \leq f(x) \Longrightarrow f(x) \in R(Y)$

$\Longrightarrow x \in f^{-1}(R(Y)) = \ker(pof)$

$\Longrightarrow R(x) \subseteq \ker(pof)$

Hence by Prop. 1.21 there exists a unique map

$\bar{f}: X/R(X) \longrightarrow Y/R(Y)$ s.t. the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "p"] & Y/R(Y) \\ X/R(X) \arrow[urr, dashed, "\bar{f}"'] & & \end{tikzcd}$$

is commutative

Now we can define a functor

$\bar{F}: \mathcal{B}_{CI} \longrightarrow \mathcal{P}_s$ s.t.

$\bar{F}(X) = X/R(X)$

and for any morphism $f: X \longrightarrow Y$ in $\mathcal{B}_{CI}$

$\bar{F}(f) = \bar{f}: X/R(X) \longrightarrow Y/R(Y)$ given by

$\bar{f}(\bar{x}) = \overline{f(x)}$

The commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "g"] \arrow[d, "p"'] & Z \arrow[d, "p"'] \\ X/R(X) \arrow[r, "\bar{f}"] & Y/R(Y) \arrow[r, "\bar{g}"] & Z/R(Z) \end{tikzcd}$$

implies that the diagram

$$\begin{array}{ccc} X & \xrightarrow{gof} & Z \\ p\downarrow & & \downarrow p \\ X/R(X) & \xrightarrow{\overline{gof}} & Z/R(Z) \end{array}$$

is commutative.

Thus we have $\overline{gof} = \bar{g}o\bar{f}$

Trivially for an identity map $I: X \longrightarrow X$ in $\mathcal{B}_{CI}$ $\bar{I}(\bar{x}) = (\bar{x})$ is an identity morphism in $\mathcal{P}_s$ i.e. ${}^{F}I_X = {}^{I}F(_X)$

Hence $F: \mathcal{B}_{CI} \longrightarrow \mathcal{P}_s$ defines as covariant functor.

Let X be a p-semi-simple algebra. If we define

$x+y = x*(0*y)$,

then $\langle X, +, 0\rangle$ is an abelian group and conversely any abelian group is a p-semi-simple algebra under the operation defined as

$x*y = x-y$

Lemma 5.8: Every BCI-homomorphism is a group homomorphism.

Proof: Let $f: X \longrightarrow Y$ be a BCI-homomorphism. Then

$$\begin{aligned} f(x_1+x_2) &= f(x_1*(0*x_2)) \\ &= f(x_1)*f(0*x_2) \\ &= f(x_1) * (f(0)*f(x_2)) \\ &= f(x_1) * (0*f(x_2)) \\ &= f(x_1) + f(x_2) \end{aligned}$$

$$\Longrightarrow f(x_1+x_2) = f(x_1) + f(x_2).$$

Lemma 5.9: Every group homomorphism is a BCI-homomorphism.

Proof: Let $f: X \longrightarrow Y$ be a group homomorphism then for any $x_1, x_2 \in X$;

$$\begin{aligned} f(x_1 * x_2) &= f(x_1 - x_2) \\ &= f(x_1) + f(-x_2) \\ &= f(x_1) - f(x_2) \\ &= f(x_1) * f(x_2) \end{aligned}$$

$$\Longrightarrow f(x_1 * x_2) = f(x_1) * f(x_2).$$

Thus we can define a functor

$$G: \mathcal{P}_s \longrightarrow \mathcal{A}b \text{ s.t.}$$

$$G(X) = X$$

and for every morphism $f: X \longrightarrow Y$ in $\mathcal{P}_s$

$G(f) = f: X \longrightarrow Y$ in $\mathcal{A}b$.

We can also construct a functor

$$H: \mathcal{A}b \longrightarrow \mathcal{P}_s \quad \text{s.t.}$$

$$H(X) = X$$

and for any group homomorphism $f: X \longrightarrow Y$ in $\mathcal{A}b$

$$H(f) = f: X \longrightarrow Y \text{ in } \mathcal{P}_s$$

Lemma 5.10: Let X be a BCI-algebra and Y a p-semi-simple BCI-algebra then Hom(X,Y) is a p-semi simple BCI-algebra.

Proof: Let $f \in R(\mathrm{Hom}(X,Y))$ then $0 \leq f \Longrightarrow 0 \leq f(x)\ \forall x \in X \Longrightarrow f(x) \in R(Y)$. But Y is a p-semi simple BCI-algebra $\Longrightarrow R(Y) = 0 \Longrightarrow f(x) = 0\ \forall x \in X \Longrightarrow f = 0 \Longrightarrow R(\mathrm{Hom}(X,Y))=0$. Thus we have Hom(X,Y) is a p-semi simple BCI-algebra.

For any BCI-algebra X we can define a covariant functor

$\mathrm{Hom}(X,-): \mathcal{P}_s \longrightarrow \mathcal{P}_s$ s.t.

$\mathrm{Hom}(X,-)(Y) = \mathrm{Hom}(X,Y)$

and $\mathrm{Hom}(X,-)(f) = \mathrm{Hom}(X,f)$

For any p-semi-simple BCI-algebra Y, we can also define a contravariant functor.

$\mathrm{Hom}(-,Y): \mathcal{B}_{CI} \longrightarrow \mathcal{P}_s$ s.t.

$\mathrm{Hom}(-,Y)(X) = \mathrm{Hom}(X,Y)$

and $\mathrm{Hom}(-,Y)(f) - \mathrm{Hom}(f,Y)$

As a consequence of the above Hom functors we can define bifunctor;

$\mathrm{Hom}(-,-): \mathcal{B}_{CI} \times \mathcal{P}_s \longrightarrow \mathcal{P}_s$ s.t.

$\mathrm{Hom}(-,-): (X,Y) = \mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(-,-)\ (f,g)=\mathrm{Hom}(f,g)$

5.9 FUNCTORS BY COMMUTATORS

Following the proposition 4.10, corollary 4.2 and 4.3 we can define a functor.

$$\mathcal{C} : \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK} \text{ s.t.}$$

$$\mathcal{C}(X) = [X,X]$$

and for every BCK-homomorphism $f: X \longrightarrow Y$

$$\mathcal{C}(f) = [f] : [X,X] \longrightarrow [Y,Y]$$

is a morphism in $\mathcal{B}_{CK}$ $\mathcal{C}$ is a covariant functor. We call this functor as *commutator ideal functor*.

Proposition 5.12: The commutator ideal functor $\mathcal{C}$ is a mono functor.

Proof: Follows from proposition 4.11.

Remarks 5.1: The commutator ideal functor, $\mathcal{C} : \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$ is a sub-ideal functor of the identity functor $I: \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$.

Proposition 5.13: The family $i = \{i_X : [X,X] \longrightarrow X \mid X \in \mathcal{B}_{CK}\}$ of morphisms defines a natural monomorphism from the functor $\mathcal{C} \longrightarrow I$.

Proof: Follows from propositions 4.11 and 4.12

With the help of Theorem 4.11 and Corollaries 4.5 and 4.6 we can define a functor

$Q : \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$ s.t.

$Q(X) = X/[X,X]$ and

for every homomorphism $f: X \longrightarrow Y$ in $\mathcal{B}_{CK}$

$Q(f) = \hat{f}: X/[X,X] \longrightarrow Y/[Y,Y]$ is a morphisms in $\mathcal{B}_{CK}$.

We call this functor as *quotient commutator functor.*

Proposition 5.14: The quotient commutator functor

$Q: X/[X,X] \longrightarrow Y/[Y,Y]$ is an epi-functor.

Proof: Follows from proposition 4.12.

Remark 5.2: The functor $Q: \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$ can be constructed as the quotient functor of the identity functor I with respect to the commutator [X,X] i.e. Q = I/C.

Proposition 5.15: The family $p = \{p_X : X \longrightarrow X/[X,X]\ X \in \mathcal{B}_{CK}\}$ defines a natural epimorphism from the identity functor I to the quotient commutator functor Q.

For a commutative BCK-algebra Z. Consider the following contravariant functor

$H(Z): \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$ defined as

$H(Z)(X) = Hom(X,Z) \ \forall X \in \mathcal{B}_{CK}$,

For every homomorphism $f: X \longrightarrow Y$ in $\mathcal{B}_{CK}$

$H(Z)(f) = Hom(f,Z): Hom(Y,Z) \longrightarrow Hom(X,Z)$,

and

$HQ(Z): \mathcal{B}_{CK} \longrightarrow \mathcal{B}_{CK}$ defined as

$HQ(Z)(X) = Hom(X/[X,X], Z) \ \forall \ X \in \mathcal{B}_{CK}$,

and for any homomorphism $f: X \longrightarrow Y$ in $\mathcal{B}_{CK}$

$HQ(Z)(f) = Hom(\hat{f},Z): Hom(Y/[Y,Y],Z) \longrightarrow Hom(X/[X,X],Z)$

Proposition 5.16: The family $\emptyset = \{\emptyset_X: Hom(X/[X,X],Z) \longrightarrow Hom(X,Z) \mid X \in \mathcal{B}_{CK}\}$ is a natural isomorphism from the functor HQ(Z) to the functor H(Z).

Proof: For any homomorphism $f: X \longrightarrow Y$ in $\mathcal{B}_{CK}$, consider the diagram

$$\begin{tikzcd} HQ(Z)(Y) \arrow[r, "HQ(Z)(f)"] \arrow[d, "\emptyset_Y"'] & HQ(Z)(X) \arrow[d, "\emptyset_X"] \\ H(Z)(Y) \arrow[r, "H(Z)(f)"] & H(Z)(X) \end{tikzcd}$$

Now for any morphism $\theta \in HQ(Z)(Y) = Hom(Y/[Y,Y],Z)$ we have

$$
\begin{aligned}
[\emptyset_X o HQ(Z)(f)](\theta) &= \emptyset_X[HQ(Z)(f)(\theta)] \\
&= \emptyset_X[\theta o \hat{f}] \\
&= (\theta o \hat{f}) o p_X \\
&= \theta o (\hat{f} o p_X) \\
&= \theta_o (p_Y o f) && \text{[By Cor.4.5]} \\
&= (\theta o p_Y) o f \\
&= HZ(f)\ (\theta o p_Y) \\
&= HZ(f)\ [\emptyset_Y(\theta)] \\
&= [HZ(f) o \emptyset_Y](\theta)
\end{aligned}
$$

since θ is arbitrary, therefore

$$\emptyset_X o HQ(Z)(f) = HZ(f) o \emptyset_Y$$

===> $\emptyset: HQ(Z) \dashrightarrow H(Z)$ is a natural transformation. Also by (iii) of Theorem 4.11 we have

$\emptyset: HQ(Z)$ ---> $H(Z)$ is an isomorphism

===> $\emptyset$ is a natural isomorphism.

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